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CONVERGENCE TO EQUILIBRIUM FOR LINEAR FOKKER-PLANCK EQUATIONS

ISABELLE TRISTANI

ABSTRACT. In this note, we investigate the spectral analysis and long time asymptotic convergence of semigroups associated to discrete, fractional and classical Fokker-Planck equations in some regime where the corresponding operators are close. We successively deal with the discrete and the classical Fokker-Planck model and the fractional and the classical Fokker-Planck model. In each case, we present results of uniform convergence to equilibrium based on perturbation and/or enlargement arguments and obtained in collaboration with S. Mischler in [7].

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1. INTRODUCTION

1.1. **The models.** In this note, we are interested in the long time asymptotic convergence of semigroups associated to some discrete, fractional and classical Fokker-Planck equations. They are simple models for describing the time evolution of a density function $f = f(t, v)$, $t \geq 0$, $v \in \mathbb{R}^d$, of particles undergoing both diffusion and (harmonic) confinement mechanisms and write

$$(1.1) \quad \partial_t f = \Lambda f = \mathcal{D}f + \operatorname{div}(vf), \quad f(0) = f_0.$$

The diffusion term may either be a discrete diffusion

$$\mathcal{D}f = \Delta_\kappa f := \kappa * f - \|\kappa\|_{L^1} f,$$

for a convenient (at least nonnegative and symmetric) kernel κ . It can also be a fractional diffusion

$$(1.2) \quad (\mathcal{D}f)(x) = -(-\Delta)^{\frac{\alpha}{2}} f(v) \\
 := c_\alpha \int_{\mathbb{R}^d} \frac{f(y) - f(v) - \chi(v-y)(v-y) \cdot \nabla f(v)}{|v-y|^{d+\alpha}} dy,$$

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with $\alpha \in (0, 2)$, $\chi \in \mathcal{D}(\mathbb{R}^d)$ a truncation function which is radially symmetric and satisfies the inequality $\mathbb{1}_{B(0,1)} \leq \chi \leq \mathbb{1}_{B(0,2)}$, and a convenient normalization constant $c_\alpha > 0$. It can finally be the classical diffusion

$$\mathcal{D}f = \Delta f := \sum_{i=1}^d \partial_{v_i}^2 f.$$

1.2. Basic properties. The main features of these equations are (expected to be) the same: they are mass preserving, namely

$$\langle f(t) \rangle = \langle f_0 \rangle, \quad \forall t \geq 0, \quad \langle f \rangle := \int_{\mathbb{R}^d} f \, dv,$$

positivity preserving, have a unique positive stationary state with unit mass and that stationary state is exponentially stable, in particular

$$(1.3) \quad f(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty,$$

for any solution associated to an initial datum f_0 with vanishing mass. Such results can be obtained using different tools as the spectral analysis of self-adjoint operators, some (generalization of) Poincaré inequalities or logarithmic Sobolev inequalities as well as the Krein-Rutman theory for positive semigroup.

The aim of this work is to initiate a kind of unified treatment of the above generalized Fokker-Planck equations and more importantly to establish that the convergence (1.3) is exponentially fast uniformly with respect to the diffusion term for a large class of initial data which are taken in a fixed weighted Lebesgue or weighted Sobolev space X .

1.3. Outline of the note. We investigate two regimes where these diffusion operators are close and for which such a uniform convergence can be established. In Section 3, we first consider the case when the diffusion operator is fractional

$$\mathcal{D}f = \mathcal{D}_\varepsilon f := -(-\Delta)^{(2-\varepsilon)/2} f, \quad \varepsilon \in (0, 2),$$

so that in the limit $\varepsilon \rightarrow 0$ we also recover the classical diffusion operator $\mathcal{D}_0 = \Delta$. In Section 4, we next consider the case when the diffusion operator is discrete

$$\mathcal{D}f = \mathcal{D}_\varepsilon f := \Delta_{\kappa_\varepsilon} f, \quad \kappa_\varepsilon := \frac{1}{\varepsilon^2} k_\varepsilon,$$

where k is a nonnegative, symmetric, normalized, smooth and decaying fast enough kernel and where we use the notation $k_\varepsilon(v) = k(v/\varepsilon)/\varepsilon^d$, $\varepsilon > 0$. In the limit $\varepsilon \rightarrow 0$, one then recovers the classical diffusion operator $\mathcal{D}_0 = \Delta$.

1.4. Main result. In order to write a rough version of our main result, we introduce some notation. We define the weighted Lebesgue space L_r^1 , $r \geq 0$, as the space of measurable functions f such that $f \langle x \rangle^r \in L^1$, where $\langle x \rangle^2 := 1 + |x|^2$. For any $f_0 \in L_r^1$, we denote as $f(t)$ the solution to the generalized Fokker-Planck equation (1.1) with initial datum $f(0) = f$ and then we define the semigroup S_Λ on X by setting $S_\Lambda(t)f := f(t)$.

Theorem 1.1 (rough version). *There exist $r > 0$ and $\varepsilon_0 \in (0, 2)$ such that for any $\varepsilon \in [0, \varepsilon_0]$, the semigroup S_{Λ_ε} is well-defined on $X := L_r^1$ and there exists a unique positive and normalized stationary solution G_ε to (1.1). Moreover, there exist $a < 0$ and $C \geq 1$ such that for any $f_0 \in X$ and any $\varepsilon \in [0, \varepsilon_0]$, there holds*

$$(1.4) \quad \|S_{\Lambda_\varepsilon}(t)f - G_\varepsilon \langle f \rangle\|_X \leq C e^{at} \|f - G_\varepsilon \langle f \rangle\|_X, \quad \forall t \geq 0.$$

Our approach is a semigroup approach in the spirit of the semigroup decomposition framework introduced by Mouhot in [8] and developed subsequently in [5, 2, 10, 4, 3]. Theorem 1.1 generalizes to the discrete diffusion Fokker-Planck equation similar results obtained for the classical Fokker-Planck equation in [2, 4] (Section 4). It also makes uniform with respect to the fractional diffusion parameter the convergence results obtained for the fractional diffusion equation in [10] (Section 3). It is worth mentioning that there exists a huge literature on the long-time behaviour for the Fokker-Planck equation as well as (to a lesser extend) for the fractional Fokker-Planck equation. We refer to the references quoted in [2, 4, 10] for details. There also probably exist many papers on the discrete diffusion equation since it is strongly related to a standard random walk in \mathbb{R}^d , but we were not able to find any precise reference in this PDE context.

2. ELEMENTS OF PROOF IN AN ABSTRACT SETTING

2.1. Notations. In what follows, for some given Banach spaces $(E, \|\cdot\|_E)$ and $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$, we denote by $\mathcal{B}(E, \mathcal{E})$ the space of bounded linear operators from E to \mathcal{E} and we denote by $\|\cdot\|_{\mathcal{B}(E, \mathcal{E})}$ or $\|\cdot\|_{E \rightarrow \mathcal{E}}$ the associated operator norm. We write $\mathcal{B}(E) = \mathcal{B}(E, E)$ when $E = \mathcal{E}$. We denote by $\mathcal{C}(E, \mathcal{E})$ the space of closed unbounded linear operators from E to \mathcal{E} with dense domain, and $\mathcal{C}(E) = \mathcal{C}(E, E)$ in the case $E = \mathcal{E}$. Moreover, for a Banach space X and $\Lambda \in \mathcal{C}(X)$ we denote by $S_{\Lambda}(t)$, $t \geq 0$, its associated semigroup. An eigenvalue $\xi \in \Sigma(\Lambda)$ is said to be isolated if there exists $r > 0$ such that

$$\Sigma(\Lambda) \cap \{z \in \mathbb{C}, |z - \xi| \leq r\} = \{\xi\}.$$

In the case when ξ is an isolated eigenvalue we may define $\Pi_{\Lambda, \xi} \in \mathcal{B}(X)$ the associated spectral projector by

$$\Pi_{\Lambda, \xi} := -\frac{1}{2i\pi} \int_{|z - \xi| = r} (\Lambda - z)^{-1} dz$$

with $0 < r < r$. When moreover the so-called “algebraic eigenspace” $\mathcal{R}(\Pi_{\Lambda, \xi})$ is finite dimensional we say that ξ is a discrete eigenvalue, written as $\xi \in \Sigma_d(\Lambda)$.

2.2. Strategy of the proof. Let us explain our approach. First, we may associate a semigroup $S_{\Lambda_{\varepsilon}}$ to the evolution equation (1.1) in many Sobolev spaces, and that semigroup is mass preserving and positive. In other words, $S_{\Lambda_{\varepsilon}}$ is a Markov semigroup and it is then expected that there exists a unique positive and unit mass steady state G_{ε} to the equation (1.1). Next, we are able to establish that the semigroup $S_{\Lambda_{\varepsilon}}$ splits as

$$(2.5) \quad \begin{aligned} S_{\Lambda_{\varepsilon}} &= S_{\varepsilon}^1 + S_{\varepsilon}^2, \\ S_{\varepsilon}^1 &\approx e^{tT_{\varepsilon}}, \quad T_{\varepsilon} \text{ finite dimensional}, \quad S_{\varepsilon}^2 = \mathcal{O}(e^{at}), \quad a < 0, \end{aligned}$$

in these many weighted Sobolev spaces. The above decomposition of the semigroup is the main technical issue of the paper. It is obtained by introducing a convenient splitting

$$(2.6) \quad \Lambda_{\varepsilon} = \mathcal{A}_{\varepsilon} + \mathcal{B}_{\varepsilon}$$

where $\mathcal{B}_{\varepsilon}$ enjoys suitable dissipativity property and $\mathcal{A}_{\varepsilon}$ enjoys some suitable $\mathcal{B}_{\varepsilon}$ -power regularity. It is worth emphasizing that we are able to exhibit such a splitting with uniform (dissipativity, regularity) estimates with respect to the diffusion parameter $\varepsilon \in [0, \varepsilon_0]$ in several weighted Sobolev spaces.

As a consequence of (2.5), we may indeed apply the Krein-Rutman theory developed in [6, 3] and exhibit such a unique positive and unit mass steady state G_{ε} . Of course for the classical and fractional Fokker-Planck equations the steady state is trivially given through an explicit formula (the Krein-Rutman theory is useless in that cases). A next

direct consequence of the above spectral and semigroup decomposition (2.5) is that there is a spectral gap in the spectral set $\Sigma(\Lambda_\varepsilon)$ of the generator Λ_ε , namely

$$(2.7) \quad \lambda_\varepsilon := \sup\{\Re \xi \in \Sigma(\Lambda_\varepsilon) \setminus \{0\}\} < 0,$$

and next that an exponential trend to the equilibrium can be established, namely

$$(2.8) \quad \|S_{\Lambda_\varepsilon}(t)f\|_X \leq C_\varepsilon e^{at} \|f\|_X \quad \forall t \geq 0, \quad \forall \varepsilon \in [0, \varepsilon_0], \quad \forall a > \lambda_\varepsilon,$$

for any initial datum $f \in X$ with vanishing mass.

Our final step consists in proving that the spectral gap (2.7) and the estimate (2.8) are uniform with respect to ε , more precisely, there exists $\lambda < 0$ such that $\lambda_\varepsilon \leq \lambda$ for any $\varepsilon \in [0, \varepsilon_0]$ and C_ε can be chosen independent to $\varepsilon \in [0, \varepsilon_0]$.

2.3. Enlargement theorem. A first way to get such uniform bounds is just to have in at least one Hilbert space $E_\varepsilon \subset L^1(\mathbb{R}^d)$ the estimate

$$\forall f \in \mathcal{D}(\mathbb{R}^d), \quad \langle f \rangle = 0, \quad (\Lambda_\varepsilon f, f)_{E_\varepsilon} \leq \lambda \|f\|_{E_\varepsilon}^2,$$

and then (2.8) essentially follows from the fact that the splitting (2.6) holds with operators which are uniformly bounded with respect to $\varepsilon \in [0, \varepsilon_0]$. It is the strategy we use in the case of the fractional diffusion (Section 3) and the work has already been made in [10] except for the simple but fundamental observation that the fractional diffusion operator is uniformly bounded (and converges to the classical diffusion operator) when it is suitable (re)scaled.

We here state the abstract enlargement theorem from [2] that we use:

Theorem 2.2. *Let E and \mathcal{E} two Banach spaces with $E \subset \mathcal{E}$ dense with continuous embedding and consider $L \in \mathcal{C}(E)$, $\mathcal{L} \in \mathcal{C}(\mathcal{E})$ with $\mathcal{L}|_E = L$. We suppose that the operators L and \mathcal{L} split as $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and $L = A + B$ with $\mathcal{A}|_E = A$ and $\mathcal{B}|_E = B$. We assume that there exist $a \in \mathbb{R}$, $n \in \mathbb{N}$ such that:*

(E1) *Localization of the spectrum of L :*

$$\Sigma(L) \cap D_a = \{0\} \subset \Sigma_d(L),$$

where $D_a := \{\xi \in \mathbb{C}, \Re \xi > a\}$; L generates a semigroup on E and $L - a$ is dissipative on $R(Id - \Pi_{L,0})$.

(E2) *Dissipativity of \mathcal{B} and boundedness of \mathcal{A} :*

$(\mathcal{B} - a)$ is hypodissipative on \mathcal{E} and $\mathcal{A} \in \mathcal{B}(\mathcal{E})$, $A \in \mathcal{B}(E)$.

(E3) *Regularizing properties of $T_n(t) := (\mathcal{A}S_{\mathcal{B}}(t))^{(n)}$:*

$$\|T_n(t)\|_{\mathcal{B}(E,E)} \leq C_{a,n} e^{at}.$$

Then, the spectrum $\Sigma(\mathcal{L})$ satisfies in \mathcal{E} the separation property:

$$\Sigma(\mathcal{L}) \cap D_a = \{0\} \subset \Sigma_d(L).$$

Moreover, for any initial datum $f \in \mathcal{E}$ and any $a > a$, we have the following estimate:

$$\forall t \geq 0, \quad \left\| S_L(t)f - \Pi_{L,0} S_L(t)f \right\|_E \leq C_a e^{a t} \|f - \Pi_{L,0} f\|_E.$$

2.4. Perturbative theorem. A second way to get the desired uniform estimate is to use a perturbation argument. Observing that, in the discrete case (Section 4),

$$\forall \varepsilon \in [0, \varepsilon_0], \quad \Lambda_\varepsilon - \Lambda_0 = \mathcal{O}(\varepsilon),$$

for a suitable operator norm, we are able to deduce that $\varepsilon \mapsto \lambda_\varepsilon$ is a continuous function at $\varepsilon = 0$, from which we readily conclude. We use here again that the considered model converges to the classical Fokker-Planck equation. In other words, the discrete model can be seen as (singular) a perturbation of the limit equation and our analyze takes advantage of such a property in order to capture the asymptotic behaviour of the related spectral objects (spectrum, spectral projector) and to conclude to the above uniform spectral decomposition. This kind of perturbative method has been introduced in [5] and improved in [9]. We here give a new and improved version of the abstract perturbation argument where some dissipativity assumptions are relaxed with respect to [9] and only required to be satisfied on the limit operator ($\varepsilon = 0$). Here is a version of this perturbative theorem in an abstract setting:

Theorem 2.3. *We consider $(\Lambda_\varepsilon)_{\varepsilon \geq 0}$ a family of operators which split into two parts as $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ and three Banach spaces $X_1 \subset X_0 \subset X_{-1}$ with continuous and dense embeddings. We suppose that exist $a \in \mathbb{R}$, $n \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that the following assumptions are satisfied:*

(P1) *Localization of spectrum of Λ_0 : in X_0 and X_1 , we have*

$$\Sigma(\Lambda_0) \cap D_a = \{0\} \subset \Sigma_d(\Lambda_0).$$

(P2) *Dissipativity of \mathcal{B}_ε and boundedness of \mathcal{A}_ε :*

for any $\varepsilon \in [0, \varepsilon_0]$, $\mathcal{B}_\varepsilon - a$ is hypodissipative in X_0 , $\mathcal{B}_0 - a$ is hypodissipative in X_j and $\mathcal{A}_\varepsilon \in \mathcal{B}(X_j)$ for $j = -1, 0, 1$.

(P3) *Regularizing properties of $T_n(t) := (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(t))^{(n)}$: for any $\varepsilon \in [0, \varepsilon_0]$, T_n satisfies for $j = -1, 0$:*

$$\|T_n(t)\|_{\mathcal{B}(X_j, X_{j+1})} dt \leq C_{a,n} e^{at}.$$

(P4) *Estimate on $\Lambda_\varepsilon - \Lambda_0$: for $j = 0, 1$,*

$$\|\mathcal{A}_\varepsilon - \mathcal{A}_0\|_{\mathcal{B}(X_j, X_{j-1})} + \|\mathcal{B}_\varepsilon - \mathcal{B}_0\|_{\mathcal{B}(X_j, X_{j-1})} \leq \eta_1(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Then, there exist $\varepsilon_1 \in (0, \varepsilon_0]$ and $\eta_2(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ such that for any $\varepsilon \in (0, \varepsilon_1]$,

$$\Sigma(\Lambda_\varepsilon) \cap D_a = \{\xi_1^\varepsilon, \dots, \xi_k^\varepsilon\} \subset \Sigma_d(\Lambda_\varepsilon);$$

$$\forall 1 \leq j \leq k, \quad |\xi_j^\varepsilon| \leq \eta_2(\varepsilon).$$

$$\dim \mathbf{R}(\Pi_{\Lambda_\varepsilon, \xi_1^\varepsilon} + \dots + \Pi_{\Lambda_\varepsilon, \xi_k^\varepsilon}) = \dim \mathbf{R}(\Pi_{\Lambda_0, 0}).$$

Moreover, for any $a \in (a, \infty) \setminus \{\Re \xi_1^\varepsilon, \dots, \Re \xi_k^\varepsilon\}$ and for any $f \in X_0$, we have for any $t \geq 0$:

$$\left\| S_{\Lambda_\varepsilon}(t)f - \sum_{j=1}^k S_{\Lambda_\varepsilon}(t)\Pi_{\Lambda_\varepsilon, \xi_j^\varepsilon} f \right\|_{X_0} \leq C_a e^{at} \left\| f - \sum_{j=1}^k \Pi_{\Lambda_\varepsilon, \xi_j^\varepsilon} f \right\|_{X_0}.$$

3. FRACTIONAL AND CLASSICAL FOKKER-PLANCK EQUATIONS

3.1. The equations and the result. In this part, for sake of simplicity, we denote $\alpha := 2 - \varepsilon \in (0, 2]$ and we deal with the equations

$$(3.9) \quad \begin{cases} \partial_t f = -(-\Delta)^{\alpha/2} f + \operatorname{div}(vf) = \Lambda_{2-\alpha} f =: \mathcal{L}_\alpha f, & \alpha \in (0, 2) \\ \partial_t f = \Delta f + \operatorname{div}(vf) = \Lambda_0 f =: \mathcal{L}_2 f. \end{cases}$$

We here recall that the fractional Laplacian $\Delta^{\alpha/2} f$ is defined for a Schwartz function f through the integral formula (1.2). Moreover, the constant c_α in (1.2) is chosen such that

$$\frac{c_\alpha}{2} \int_{|z| \geq 1} \frac{z_1^2}{|z|^{d+\alpha}} = 1,$$

which implies that $c_\alpha \approx (2 - \alpha)$. By duality, we can extend the definition of the fractional Laplacian to the following class of functions:

$$\left\{ f \in L^1_{\text{loc}}(\mathbb{R}^d), \int_{\mathbb{R}^d} |f(x)| \langle x \rangle^{-d-\alpha} dx < \infty \right\}.$$

In particular, one can define $(-\Delta)^{\alpha/2} m$ when $q < \alpha$.

In this part, we would like to apply an enlargement argument (Theorem 2.2) to this family of equations (3.9) and obtain a result uniform in $\alpha \in [\alpha_0, 2]$. To do that, it is necessary to be able to check that the assumptions of Theorem 2.2 are fulfilled uniformly in α .

We recall that the equation $\partial_t f = \mathcal{L}_\alpha f$ admits a unique equilibrium of mass 1 that we denote G_α (see [1] for the case $\alpha < 2$). Moreover, if $\alpha < 2$, one can prove the following estimate $G_\alpha(x) \approx \langle x \rangle^{-d-\alpha}$ (see [10]) and for $\alpha = 2$, we have an explicit formula $G_2(x) = (2\pi)^{-d/2} e^{-|x|^2/2}$. The main result of this section reads:

Theorem 3.4. *Assume $\alpha_0 \in (0, 2)$ and $q < \alpha_0$. There exists an explicit constant $a_0 < 0$ such that for any $\alpha \in [\alpha_0, 2]$, the semigroup $S_{L_\alpha}(t)$ associated to the fractional Fokker-Planck equation (3.9) satisfies: for any $f \in L^1_q$, any $a > a_0$ and any $\alpha \in [\alpha_0, 2]$,*

$$\|S_{L_\alpha}(t)f - G_\alpha \langle f \rangle\|_{L^1_q} \leq C_a e^{at} \|f - G_\alpha \langle f \rangle\|_{L^1_q}$$

for some explicit constant $C_a \geq 1$. In particular, the spectrum $\Sigma(\mathcal{L}_\alpha)$ of \mathcal{L}_α satisfies the separation property $\Sigma(\mathcal{L}_\alpha) \cap D_{a_0} = \{0\}$ in L^1_q for any $\alpha \in [\alpha_0, 2]$.

3.2. Idea of the proof. Let us consider $\alpha_0 \in (0, 2)$ and $q < \alpha_0$. For sake of simplicity, we denote $m(v) := \langle v \rangle^q$ and $\Delta_\alpha := -(-\Delta)^{\alpha/2}$. The spaces that we are going to consider in view of applying the enlargement theorem are:

$$E_\alpha := L^2(G_\alpha^{-1/2}) \subset \mathcal{E} := L^1(m).$$

The goal is to apply the enlargement theorem (Theorem 2.2) for each $\alpha \in [\alpha_0, 2]$. To do that, we introduce the following splitting of the operator \mathcal{L}_α :

$$\mathcal{A}_\alpha f := M \chi_R f, \quad \mathcal{B}_\alpha f := \mathcal{L}_\alpha f - \mathcal{A}_\alpha f$$

where $\chi_R \in \mathcal{D}(\mathbb{R}^d)$ satisfies $\mathbf{1}_{B(0,R)} \leq \chi_R \leq \mathbf{1}_{B(0,2R)}$ for some constants $M, R \in \mathbb{R}^+$ to be determined later.

3.2.1. *Check of (E1).* We recall a result from [1] which establishes an exponential decay to equilibrium for the semigroup $S_{L_\alpha}(t)$ in the small space $L^2(G_\alpha^{-1/2})$.

Theorem 3.5. *There exists an explicit constant $\lambda > 0$ such that for any $\alpha \in (0, 2)$,*

- (1) *in $E_\alpha = L^2(G_\alpha^{-1/2})$, there holds $\Sigma(\mathcal{L}_\alpha) \cap D_{-\lambda} = \{0\}$;*
- (2) *the following estimate holds: for any $f \in E_\alpha$ and any $a > -\lambda$,*

$$\|S_{L_\alpha}(t)f - G_\alpha\langle f \rangle\|_{E_\alpha} \leq e^{at} \|f - G_\alpha\langle f \rangle\|_{E_\alpha}, \quad \forall t \geq 0.$$

3.2.2. *Check of (E2).* We first recall some elements on the notion of dissipativity. An operator $\Lambda - a$ is said to be *dissipative* in $(X, \|\cdot\|_X)$ if for any $f \in D(\Lambda)$, there exists $\phi \in F(f) := \{\phi \in X, \langle f, \phi \rangle = \|f\|_X^2 = \|\phi\|_X^2\}$ such that $\Re e\langle (\Lambda - a)f, \phi \rangle \leq 0$. We do not enter into details concerning the notion of hypodissipativity but one can keep in mind that hypodissipativity is nothing but dissipativity for an equivalent norm.

In practice, there is no need to go back to the definition given above. It is enough to estimate integrals to prove that an operator is dissipative. For example, if we want to prove that $\Lambda - a$ is dissipative in $X = L^1(m)$, showing the inequality:

$$\forall f \in D(\Lambda), \quad \int_{\mathbb{R}^d} (\Lambda f) \operatorname{sign}(f) m \leq a \int_{\mathbb{R}^d} |f| m$$

is enough to conclude. Indeed, if we consider $f \in D(\Lambda)$, one can define $\varphi_f \in X$ by

$$\langle \varphi_f, h \rangle = \left(\int_{\mathbb{R}^d} h \operatorname{sign}(f) m \right) \|f\|_{L^1(m)}, \quad \forall h \in X$$

and both check that $\varphi_f \in F(f)$ and $\Re e\langle (\Lambda - a)f, \varphi_f \rangle \leq 0$.

Let us now go back to our operator \mathcal{L}_α . Before going into the proof of the dissipativity of \mathcal{B}_α , one can notice that $\mathcal{A}_\alpha \in \mathcal{B}(E)$ and $\mathcal{B}(\mathcal{E})$ since \mathcal{A}_α is just a truncation operator. To get the dissipativity properties of \mathcal{B}_α , we first estimate the integral $\int_{\mathbb{R}^d} (\mathcal{L}_\alpha f) \operatorname{sign}(f) m$. This proof is an adaptation of the proof of Lemma 5.1 from [10] taking into account the constant c_α . Indeed, we have

$$\int_{\mathbb{R}^d} (\mathcal{L}_\alpha f) \operatorname{sign} f m \leq \int_{\mathbb{R}^d} |f| m \left(\frac{\Delta_\alpha(m)}{m} - \frac{v \cdot \nabla m}{m} \right).$$

We can then show that thanks to the rescaling constant c_α , $\Delta_\alpha(m)/m$ goes to 0 at infinity uniformly in $\alpha \in [\alpha_0, 2)$. As a consequence, if $a > -q$, since $(v \cdot \nabla m)/m$ goes to $-q$ at infinity, one may choose M and R such that for any $\alpha \in [\alpha_0, 2)$,

$$\frac{\Delta_\alpha(m)}{m} - \frac{v \cdot \nabla m}{m} - M \chi_R \leq a, \quad \text{on } \mathbb{R}^d,$$

which gives us the dissipativity of \mathcal{B}_α uniformly in α in \mathcal{E} .

Finally, up to change the constants M and R , if $a > \min(-\lambda, -q)$ (where λ is defined in Theorem 3.5), one can check that $\mathcal{B}_\alpha - a$ is also dissipative in E_α using the dissipativity property of \mathcal{L}_α coming from Theorem 3.5.

3.2.3. *Check of (E3).* We now want to investigate the regularization properties of the semigroup $\mathcal{A}_\alpha S_{E_\alpha}(t)$. More precisely, we want to prove that taking the n -convolution of this semigroup allows us to go from the large space \mathcal{E} into the small one E_α . Let us notice that we can get rid of the weights thanks to the truncation operator \mathcal{A}_α . It remains to

prove that the semigroup $S_{B_\alpha}(t)$ regularizes from L^1 to L^2 . The key argument to get this kind of property is the *fractional Nash inequality*:

$$\|f\|_{L^2} \leq C \|f\|_{L^1}^{\alpha/(d+\alpha)} \|f\|_{\dot{H}^{\alpha/2}}^{d/(d+\alpha)}, \quad \forall f \in L^1(\mathbb{R}^d) \cap H^{\alpha/2}(\mathbb{R}^d).$$

The following computations allow to understand the role of Nash inequality. If we consider f_t solution of $\partial_t f_t = \mathcal{B}_\alpha f_t$, $f_0 = f$, we can compute

$$\frac{1}{2} \frac{d}{dt} \|S_{B_\alpha}(t)f\|_{L^2}^2 = \int_{\mathbb{R}^d} (\mathcal{B}_\alpha f_t) f_t.$$

It is thus crucial to estimate the integral $\int_{\mathbb{R}^d} (\mathcal{B}_\alpha f) f$. We have:

$$\begin{aligned} \int_{\mathbb{R}^d} (\mathcal{B}_\alpha f) f &\leq -\frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (f(v) - f(y))^2 |v - y|^{-d-\alpha} dy dv + b \int_{\mathbb{R}^d} f^2 \\ (3.10) \quad &\leq -\frac{1}{2} \|f\|_{\dot{H}^{\alpha/2}}^2 + b \|f\|_{L^2}^2, \\ &\leq -C \|f\|_{L^2}^{2(d+\alpha)/d} \|f\|_{L^1}^{-(2\alpha)/d} + b \|f\|_{L^2}^2, \quad b \in \mathbb{R}^+. \end{aligned}$$

Consequently, we have a non positive term which is going to induce a gain of regularity. Thanks to this inequality, we can obtain a differential inequality on $f_t = S_{B_\alpha}(t)f$ and obtain:

$$\|S_{B_\alpha}(t)f\|_{L^2} \leq C \frac{e^{bt}}{t^{d/(2\alpha)}} \|f\|_{L^1}, \quad \forall t \geq 0.$$

One can notice that we have the gain of regularity that we wanted. However, the rate in the previous equality is not the one that we expected. But, thanks to a trick developed in [2, 4], one can recover the wanted rate taking the n -convolution of the semigroup, for n large enough. As a conclusion, one can obtain that there exists $n \in \mathbb{N}$ such that

$$\|(\mathcal{A}_\alpha S_{B_\alpha}(t))^{(n)}\|_{\mathcal{B}(\mathcal{E}, E_\alpha)} \leq C e^{at}, \quad \forall t \geq 0$$

which concludes this part.

3.2.4. Conclusion. In summary, we have checked that assumptions (E1), (E2) and (E3) are fulfilled for any $\alpha \in [\alpha_0, 2]$. We can thus apply the enlargement theorem (Theorem 2.2) for each α with the spaces $E_\alpha \subset L^1(\langle v \rangle^q)$ with $q < \alpha_0$. The fact that the assumptions are satisfied uniformly in α implies that we obtain a rate of decrease for the semigroup $S_{L_\alpha}(t)$ which is uniform with respect to α in Theorem 3.4.

4. DISCRETE AND CLASSICAL FOKKER-PLANCK EQUATIONS

4.1. The equations and the result. In this section, we consider a kernel $k \in W^{2,1}(\mathbb{R}^d) \cap L^1_3(\mathbb{R}^d)$ which is symmetric, i.e. $k(-v) = k(v)$ for any $v \in \mathbb{R}^d$, satisfies the normalization condition

$$(4.11) \quad \int_{\mathbb{R}^d} k(v) \begin{pmatrix} 1 \\ v \\ v \otimes v \end{pmatrix} dv = \begin{pmatrix} 1 \\ 0 \\ 2I_d \end{pmatrix},$$

as well as the positivity condition: there exist $\kappa_0, \rho > 0$ such that

$$(4.12) \quad k \geq \kappa_0 \mathbf{1}_{B(0, \rho)}.$$

We define $k_\varepsilon(v) := 1/\varepsilon^d k(v/\varepsilon)$, $v \in \mathbb{R}^d$ for $\varepsilon > 0$, and we consider the discrete and classical Fokker-Planck equations

$$(4.13) \quad \begin{cases} \partial_t f = \frac{1}{\varepsilon^2} (k_\varepsilon * f - f) + \operatorname{div}(vf) =: \Lambda_\varepsilon f, & \varepsilon > 0, \\ \partial_t f = \Delta f + \operatorname{div}(vf) =: \Lambda_0 f. \end{cases}$$

The goal of this part is to get a result of convergence to the equilibrium uniform with respect to the parameter ε . To do that, we are going to combine the two theorems exposed in the abstract section (Section 2). First, we apply the perturbative one (Theorem 2.3) and then, in order to get a result of convergence in a weighted L^1 space, we conclude by using the enlargement theorem (Theorem 2.2).

The main result of the section reads as follows.

Theorem 4.6. *Let us assume that $r > d/2$ and consider a symmetric kernel k belonging to $W^{2,1}(\mathbb{R}^d) \cap L^1_{2r_0+3}$ where $r_0 > \max(r + d/2, 5 + d/2)$ which satisfies (4.11) and (4.12).*

(1) *For any $\varepsilon > 0$, there exists a positive and unit mass normalized steady state G_ε in $L^1_r(\mathbb{R}^d)$ to the discrete Fokker-Planck equation (4.13).*

(2) *There exist explicit constants $a_0 < 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$, the semigroup $S_{\Lambda_\varepsilon}(t)$ associated to the discrete Fokker-Planck equation (4.13) satisfies: for any $f \in L^1_r$ and any $a > a_0$,*

$$\|S_{\Lambda_\varepsilon}(t)f - G_\varepsilon \langle f \rangle\|_{L^1_r} \leq C_a e^{at} \|f - G_\varepsilon \langle f \rangle\|_{L^1_r}, \quad \forall t \geq 0,$$

for some explicit constant $C_a \geq 1$. In particular, the spectrum $\Sigma(\Lambda_\varepsilon)$ of Λ_ε satisfies the separation property $\Sigma(\Lambda_\varepsilon) \cap D_{a_0} = \{0\}$ in L^1_r , where we recall that D_α is defined through $D_\alpha = \{\xi \in \mathbb{R}^d; \Re \xi > \alpha\}$.

4.2. Idea of the proof. We recall that $\chi \in \mathcal{D}(\mathbb{R}^d)$ is a truncation function radially symmetric and satisfying $\mathbf{1}_{B(0,1)} \leq \chi \leq \mathbf{1}_{B(0,2)}$. We define χ_R by $\chi_R(v) := \chi(v/R)$ for $R > 0$ and we denote $\chi_R^c := 1 - \chi_R$.

For $\varepsilon > 0$, we define the splitting $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ with

$$\mathcal{A}_\varepsilon f := M \chi_R (k_\varepsilon * f),$$

$$\mathcal{B}_\varepsilon f := \left(\frac{1}{\varepsilon^2} - M \right) (k_\varepsilon * f - f) + M \chi_R^c (k_\varepsilon * f - f) + \operatorname{div}(vf) - M \chi_R f,$$

for some constants M, R to be chosen later. Similarly, we define the splitting $\Lambda_0 = \mathcal{A}_0 + \mathcal{B}_0$ with $\mathcal{A}_0 f := M \chi_R f$ and thus $\mathcal{B}_0 f := \Lambda_0 f - M \chi_R f$ for some constants M, R to be chosen later.

First step of the proof. The first step is to develop the perturbative argument coming from Theorem 2.3 with the following spaces:

$$X_1 := H^6_{r_0+1} \subset X_0 := H^3_{r_0} \subset X_{-1} := L^2_{r_0},$$

where $r_0 > d/2 + 5$, we denote $m(v) := \langle v \rangle^{r_0}$.

4.2.1. Check of (P1). The behavior of the semigroup associated to Λ_0 , the classical Fokker-Planck operator has been largely studied for a few decades by different means: Logarithmic Sobolev inequalities, Poincaré inequality and Lyapunov condition in spaces of type $L^2(e^{v^2/2})$ with a weight prescribed by the maxwellian equilibrium. More recently, improvements have been made in the sense that the space in which such a decay property holds has been enlarged thanks to the theory developed in [2] by Gualdani et al. and [4] by Mischler and Mouhot (Theorem 2.2). In particular, we have the following result:

Theorem 4.7. Consider $i \in \{-1, 0, 1\}$. There exists $a_0 < 0$ such that for any $f \in X_i$ and any $a > a_0$,

$$\|S_{\Lambda_0}(t)f - G_0\langle f \rangle\|_{X_i} \leq C_a e^{at} \|f - G_0\langle f \rangle\|_{X_i}, \quad \forall t \geq 0$$

where $S_{\Lambda_0}(t)$ is the semigroup associated to the generator Λ_0 and G_0 is the unique equilibrium of the equation of mass 1.

We thus deduce that (P1) is checked.

4.2.2. *Check of (P2).* First, we clearly have that \mathcal{A}_ε is bounded in X_i , for $i \in \{-1, 0, 1\}$ uniformly in ε . This comes from Young inequality and the fact that $\|k_\varepsilon\|_{L^1} = 1$.

Concerning the dissipativity properties of \mathcal{B}_ε , we just give an example of computation in $L^2(m)$ with $m(v) = \langle v \rangle^{r_0}$. We split the operator in several pieces

$$\begin{aligned} \mathcal{B}_\varepsilon f &= \left(\frac{1}{\varepsilon^2} - M \right) (k_\varepsilon * f - f) + M \chi_R^c (k_\varepsilon * f - f) \\ &\quad + \operatorname{div}(vf) - M \chi_R f =: \mathcal{B}_\varepsilon^1 f + \dots + \mathcal{B}_\varepsilon^4 f, \end{aligned}$$

and we estimate each term

$$T_i := \int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon^i f) f m^2$$

separately. From now on, we consider $a > d/2 - r_0$, we fix $\varepsilon_1 > 0$ such that $M \leq 1/(2\varepsilon_1^2)$ and we consider $\varepsilon \in (0, \varepsilon_1]$.

We first deal with T_1 . We observe that

$$(4.14) \quad (f(y) - f(v)) f(v) = -\frac{1}{2}(f(y) - f(v))^2 + \frac{1}{2}(f^2(y) - f^2(v)).$$

We then compute

$$\begin{aligned} T_1 &= \left(\frac{1}{\varepsilon^2} - M \right) \int_{\mathbb{R}^d \times \mathbb{R}^d} k_\varepsilon(v - y) (f(y) - f(v)) f(v) m^2(v) dy dv \\ &\leq \frac{1}{2} \left(\frac{1}{\varepsilon^2} - M \right) \int_{\mathbb{R}^d \times \mathbb{R}^d} (f^2(y) - f^2(v)) k_\varepsilon(v - y) m^2(v) dy dv \\ &= \frac{1}{2} \left(\frac{1}{\varepsilon^2} - M \right) \int_{\mathbb{R}^d \times \mathbb{R}^d} (m^2(y) - m^2(v)) k_\varepsilon(v - y) f^2(v) dy dv, \end{aligned}$$

where we have performed a change of variables to get the last equality. From a Taylor expansion, we have

$$m^2(y) - m^2(v) = (y - v) \cdot \nabla m^2(v) + \Theta(v, y),$$

where

$$\begin{aligned} |\Theta(v, y)| &\leq \frac{1}{2} \int_0^1 |D^2 m^2(v + \theta(y - v))(y - v, y - v)| d\theta \\ &\leq C |v - y|^2 \langle v \rangle^{2r_0 - 2} \langle v - y \rangle^{2r_0 - 2}, \end{aligned}$$

for some constant $C \in (0, \infty)$. The term involving the gradient of m^2 gives no contribution because of (4.11) and we thus obtain

$$\begin{aligned} (4.15) \quad T_1 &\leq C (1 - M\varepsilon^2) \int_{\mathbb{R}^d \times \mathbb{R}^d} k_\varepsilon(v - y) \frac{|v - y|^2}{\varepsilon^2} \langle v - y \rangle^{2r_0 - 2} dy f^2(v) \langle v \rangle^{2r_0 - 2} dv \\ &\leq C \int_{\mathbb{R}^d} f^2(v) \langle v \rangle^{2r_0 - 2} dv. \end{aligned}$$

We now treat the second term T_2 . Proceeding as above and thanks to (4.14) again, we have

$$\begin{aligned} T_2 &= \int_{\mathbb{R}^d \times \mathbb{R}^d} M \chi_R^c(v) k_\varepsilon(v-y) (f(y) - f(v)) f(v) m^2(v) dy dv \\ &\leq \frac{M}{p} \int_{\mathbb{R}^d \times \mathbb{R}^d} k(z) \{ \chi_R^c(v+\varepsilon z) m^2(v+\varepsilon z) - \chi_R^c(v) m^2(v) \} dz f^2(v) dv \end{aligned}$$

Using the mean value theorem

$$\chi_R^c(v+\varepsilon z) = \chi_R^c(v) + \varepsilon z \cdot \nabla \chi_R^c(v + \theta \varepsilon z), \quad m^2(v+\varepsilon z) = m^2(v) + \varepsilon z \cdot \nabla m^2(v + \theta \varepsilon z),$$

for some $\theta, \theta \in (0, 1)$, and the estimates

$$|\nabla \chi_R^c| \leq C_R \quad \text{and} \quad |\nabla m^2(v + \theta \varepsilon z)| \leq C \langle v \rangle^{2q-1} \langle z \rangle^{2q-1},$$

we conclude that

$$(4.16) \quad T_2 \leq M C_R \varepsilon \int_{\mathbb{R}^d} f^2 m^2.$$

As far as T_3 is concerned, we just perform an integration by parts:

$$\begin{aligned} (4.17) \quad T_3 &= d \int_{\mathbb{R}^d} f^2 m^2 - \frac{1}{2} \int_{\mathbb{R}^d} f^2 \operatorname{div}(v m^2) \\ &= \int_{\mathbb{R}^d} f^2(v) m^2(v) \left(\frac{d}{2} - \frac{r_0 |v|^2}{\langle v \rangle^2} \right) dv. \end{aligned}$$

The estimates (4.15), (4.16) and (4.17) together give

$$\begin{aligned} \int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon f) f m^2 &\leq \int_{\mathbb{R}^d} f^2 m^2 \left(C \langle v \rangle^{-2} + \frac{d}{2} - \frac{r_0 |v|^2}{\langle v \rangle^2} + M C_R \varepsilon - M \chi_R \right) \\ &= \int_{\mathbb{R}^d} f^2 m^2 (\psi_R^\varepsilon - M \chi_R), \end{aligned}$$

where we have denoted

$$(4.18) \quad \psi_R^\varepsilon(v) := C \langle v \rangle^{-2} + \frac{d}{2} - \frac{r_0 |v|^2}{\langle v \rangle^2} + M C_R \varepsilon.$$

Because $\psi_R^\varepsilon(v) \rightarrow d/2 - r_0$ when $\varepsilon \rightarrow 0$ and $|v| \rightarrow \infty$, and $a > d/2 - r_0$, we can choose $M \geq 0$, $R \geq 0$ and $\varepsilon_0 \leq \varepsilon_1$ such that for any $\varepsilon \in (0, \varepsilon_0]$,

$$\forall v \in \mathbb{R}^d, \quad \psi_R^\varepsilon(v) \leq a.$$

As a conclusion, for such a choice of constants, we obtain the uniform dissipativity of \mathcal{B}_ε in $L_{r_0}^2$. We refer to [2, 4] for the proof in the case $\varepsilon = 0$.

We do not enter into details concerning the dissipativity properties of \mathcal{B}_ε in higher Sobolev norms of kind $H_{r_0}^s$ with $s \geq 1$. The notion of hypodissipativity is here necessary. Indeed, we are not able to prove that \mathcal{B}_ε is dissipative in $H_{r_0}^s$ for $s \geq 1$. However, we can prove such a property for an equivalent norm to the usual H^s norm, which corresponds with the notion of hypodissipativity.

4.2.3. *Check of (P3).* We here need to prove that the semigroup $\mathcal{A}_\varepsilon S_{B_\varepsilon}(t)$ regularizes from L^2 to H^1 . Indeed, if we are able to do that, if we take n large enough, we'll have the desired result: $(\mathcal{A}_\varepsilon S_{B_\varepsilon})^{(n)}(t)$ regularizes from X_i to X_{i+1} , $i = -1, 0$.

To get such a property, we are going to exploit the non positive term coming from the decomposition (4.14): we consider f_t solution of

$$\partial_t f_t = \mathcal{B}_\varepsilon f_t, \quad f_0 = f.$$

Then

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|f_t\|_{L^2(m)}^2 &= \int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon f_t) f_t m^2 \\ &\leq -\frac{1}{4\varepsilon^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (f_t(y) - f_t(v))^2 k_\varepsilon(v - y) dy dv + a \|f_t\|_{L^2(m)}^2. \end{aligned}$$

One can notice that this computation is similar to the one done for the fractional Fokker-Planck equation (3.10). However, in the latter case, it was easy to identify the non positive term (a homogeneous fractional Sobolev norm). In our case, it is not clear that the non positive term induces a gain of regularity. The key estimate, which is obtained thanks to the assumptions made on the kernel k , is the following:

$$\|k_\varepsilon * v f\|_{H^1}^2 \leq \frac{K}{\varepsilon^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (f(y) - f(v))^2 k_\varepsilon(v - y) dy dv.$$

We can then prove that:

$$\int_0^t \|k_\varepsilon * v f_s\|_{H^1}^2 e^{-2as} ds \leq -\frac{1}{2a} \|f\|_{L^2(m)}^2, \quad \forall t \geq 0,$$

and then

$$\int_0^t \|\mathcal{A}_\varepsilon S_{B_\varepsilon}(s) f\|_{H^1(m)} e^{-as/2} ds \leq C \|f\|_{L^2(m)}.$$

If we take the 2-convolution of the semigroup $\mathcal{A}_\varepsilon S_{B_\varepsilon}(t)$, we are able to recover a pointwise of type of (P3). Then, taking the n convolution of the semigroup $\mathcal{A}_\varepsilon S_{B_\varepsilon}(t)$ for n large enough allows us to recover the good rate in the last inequality and prove that $(\mathcal{A}_\varepsilon S_{B_\varepsilon})^{(n)}(t)$ regularizes from X_i to X_{i+1} for $i = -1, 0$.

4.2.4. *Check of (P4).* The proof of the convergences for $s \in \mathbb{N}$

$$\|\mathcal{A}_\varepsilon - \mathcal{A}_0\|_{\mathcal{B}(H^{s+1}(m), H^s(m))} \xrightarrow[\varepsilon]{0} 0 \quad \text{and} \quad \|\mathcal{B}_\varepsilon - \mathcal{B}_0\|_{\mathcal{B}(H^{s+3}(m), H^s(m))} \xrightarrow[\varepsilon]{0} 0$$

is quasi immediate. Let us just mention that we need to perform a Taylor expansion to prove the second one.

4.2.5. *Krein-Rutman argument.* Before going into the conclusion given by the perturbative argument, let us underline the fact that the semigroup $S_{\Lambda_\varepsilon}(t)$ is a positive semigroup. Moreover, we can prove that it satisfies a strong maximum principle thanks to (4.12). We can thus use the Krein-Rutman theory revisited in [6] which implies that for any $\varepsilon > 0$, there exists a unique $G_\varepsilon > 0$ such that $\|G_\varepsilon\|_{L^1} = 1$, $\Lambda_\varepsilon G_\varepsilon = 0$ and $\Pi_\varepsilon f = \langle f \rangle G_\varepsilon$. It also implies that for any $\varepsilon > 0$, there exists $a_\varepsilon < 0$ such that in $X = L_r^1$ or $X = H_{r_0}^s$ for any $s \in \mathbb{N}$, there holds

$$\Sigma(\Lambda_\varepsilon) \cap D_{a_\varepsilon} = \{0\}$$

and

$$(4.19) \quad \forall t \geq 0, \quad \|S_{\Lambda_\varepsilon}(t) f - \langle f \rangle G_\varepsilon\|_X \leq e^{at} \|f - \langle f \rangle G_\varepsilon\|_X, \quad \forall a > a_\varepsilon.$$

4.2.6. *Conclusion 1.* Combining this argument with the perturbative one, we are thus able to prove the following proposition:

Proposition 4.8. *There exist $a_0 < 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$, the following properties hold in $X_0 = H_{r_0}^3$:*

- (1) $\Sigma(\Lambda_\varepsilon) \cap D_{a_0} = \{0\}$;
- (2) for any $f \in X_0$ and any $a > a_0$,

$$\|S_{\Lambda_\varepsilon}(t)f - G_\varepsilon\langle f \rangle\|_{X_0} \leq C_a e^{at} \|f - G_\varepsilon\langle f \rangle\|_{X_0}, \quad \forall t \geq 0$$

for some explicit constant $C_a > 0$.

Second step of the proof. We now want to enlarge the space in which the conclusion of the previous proposition holds. To do that, we use the enlargement Theorem 2.2. Our “small” space is $H_{r_0}^3$ and our “large” space is L_r^1 (notice that $r_0 > r + d/2$ implies the embedding $H_{r_0}^3 \subset L_r^1$) and we use exactly the same splitting as in the previous part. (E1) is satisfied from the first step of the proof. The boundedness of \mathcal{A}_ε and the dissipativity of \mathcal{B}_ε are trivially satisfied similarly as in the first step of the proof and thus (E2) is checked. However, it remains to prove that the semigroup $\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(t)$ regularizes from L^1 to L^2 to get that (E3) is checked. To do that, we use a duality argument. Performing similar computations as for the check of (P3), we show that for any $s \in \mathbb{N}$ (in particular for $s > d/2$), there exists $n \in \mathbb{N}$ such that

$$(\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(t))^{(n)} : L^2 \rightarrow H^s$$

where \mathcal{A}_ε and \mathcal{B}_ε are the formal dual operators of \mathcal{A}_ε and \mathcal{B}_ε . Consequently,

$$(S_{\mathcal{B}_\varepsilon}(t)\mathcal{A}_\varepsilon)^{(n)} : H^{-s} \rightarrow L^2.$$

We then use the Sobolev embedding $L^1 \hookrightarrow H^{-s}$ for $s > d/2$ to conclude that

$$(\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(t))^{(n+1)} : L^1 \rightarrow L^2.$$

Up to increase the value of n , we can suppose that this regularization property is satisfied with a good rate, which yields the result and conclude the proof of Theorem 4.6.

5. PERSPECTIVES

The Landau equation is a kinetic model in plasma physics and writes

$$\partial_t f = Q_L(f, f).$$

It can be obtained in grazing collision limit of the Boltzmann equation. For a good choice of rescaling of the Boltzmann collision operator B_ε , the following convergence holds

$$Q_{B_\varepsilon}(g, f) \rightarrow Q_L(g, f) \quad \text{as } \varepsilon \rightarrow 0.$$

In order to study the longtime behavior of those equations, we can study the linearized problems and consider both linearized problems in the same family:

$$\begin{cases} \partial_t f = \Lambda_\varepsilon h := Q_{B_\varepsilon}(\mu, h) + Q_{B_\varepsilon}(h, \mu), & \varepsilon > 0 \\ \partial_t f = \Lambda_0 h := Q_L(\mu, h) + Q_L(h, \mu), \end{cases}$$

where μ is the maxwellian equilibrium of both equations. The perturbative strategy used to handle the Fokker-Planck problems might be applied in this case, it would provide us a rate of convergence to equilibrium uniform with respect to ε for Boltzmann and Landau equations in a close to equilibrium regime.

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