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BOUNDS FOR KDV AND THE 1-D CUBIC NLS EQUATION IN ROUGH FUNCTION SPACES

HERBERT KOCH

Abstract. We consider the cubic Nonlinear Schrödinger Equation (NLS) and the Korteweg-de Vries equation in one space dimension. We prove that the solutions of NLS satisfy a-priori local in time H^s bounds in terms of the H^s size of the initial data for $s \geq -\frac{1}{4}$ (joint work with D. Tataru, [15, 14]) , and the solutions to KdV satisfy global a priori estimate in H^{-1} (joint work with T. Buckmaster [2]).

1. Introduction

The Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} - 6uu_x = 0 \tag{1}$$

its close relatives, the modified KdV (mKdV) equation

$$v_t + v_{xxx} - v^2 v_x = 0 \tag{2}$$

and the cubic Nonlinear Schrödinger equation (NLS)

$$iu_t - u_{xx} \pm u|u|^2 = 0, \qquad u(0) = u_0, \tag{3}$$

in one space dimension, either focusing or defocusing have rich, closely related and multiply connected structures: KdV and NLS are the most important asymptotic equations in nonlinear wave propagation. All three equations have a property called integrability. Integrable equations often come in families called hierarchies. It is interesting to note that mKdV is the second equation in the NLS hierarchy, and the flows of complex mKdV and defocusing NLS commute. On the other hand the Miura map

$$v \to v_x + v^2$$

maps solutions of mKdV to solutions of KdV.

The KdV is invariant with respect to the scaling

$$u(x,t) \to \lambda^2 u(\lambda x, \lambda^2 t)$$

as is the Sobolev space $\dot H^{-\frac{3}{2}}$ with $-3/2$ derivatives in L^2, mKdV and NLS are invariant with respect to the scaling

$$u(x,t) \to \lambda u(\lambda x, \lambda^2 t)$$

as is $\dot H^{-1/2}$, the space of functions with $-1/2$ derivatives in L^2. Solutions to suitable initial data can be constructed using the contraction mapping principle and Fourier restriction spaces in $H^{-3/4}$ (KdV, [7]), $H^{1/4}$ (mKdV) and L^2 on the real line, and in $H^{-1/2}$ (KdV), L^2(NLS) and $H^{1/2}$ (mKdV) on the one dimensional torus. The simplest case is NLS on the real line: The Strichartz estimates

$$\|v\|_{L^6} \lesssim \|v(0)\|_{L^2}$$

for solutions to $i\partial_t v - v_{xx} = 0$ imply for solutions to NLS

$$\begin{aligned}\|u\|_{L^\infty([-T,T];L^2(\mathbb{R}))} + \|u\|_{L^6((-T,T)\times\mathbb{R})} \lesssim & \|u(0)\|_{L^2} + \||u|^2u\|_{L^1([0,T],L^2)} \\ \lesssim & \|u(0)\|_{L^2} + T^{1/2}\|u\|_{L^6}^3\end{aligned}$$

which by a contraction argument yields existence of a unique solution if

$$T << \|u(0)\|_{L^2}^{-4}.$$

Global existence is easy since the L^2 norm is conserved, but control on differences is weak:

$$\|u(t) - v(t)\|_{L^2(\mathbb{R})} \le e^{ct^{1/2}\|u_0\|_{L^2}^2}\|u(0) - v(0)\|_{L^2} \tag{4}$$

is the best one gets out of the contraction argument. The argument for KdV and mKdV is considerably more involved but the structure of the argument is the same.

The statements are sharp in the following sense: The construction mapping principle combined with the implicit function theorem yields solutions depending smoothly on the initial data, and in H^s with s below this critical number the map from initial data to the solutions is known to fail to be uniformly continuous in balls. (see [13], [4]). Recently Molinet [18] has shown a much stronger failure of wellposedness for KdV in H^s with $s < -1$: The map from the initial data $L^2 \ni u_0 \to \mathcal{D}$ fails to be continuous when considered with topology of H^s to distributions $\mathcal{D}^*$. This ends the hope to obtain wellposedness results for KdV up to the critical space $H^{-3/2}$.

There are reasons to try to go beyond these wellposedness results: One would like to get a better understanding of the interaction of waves and one would like to have more information on the flow for large data and large initial data than (4) provides. A Priori estimates below the regime of the contraction mapping principle are likely to provide information for large times resp. large data.

It is remarkable that Kappeler and Topalov [11] were able to show for periodic KdV that the map from the initial data $u_0 \in L^2$ to $C(L^2)$ extends to a unique continuous map from H^{-1} to $C(I, H^{-1})$. The proof relies on inverse scattering theory and the theory of Riemann surfaces. It is optimal in view of Molinet's illposedness result [18] , and it seems to be out of reach by PDE arguments. The periodic case is more difficult to address by PDE arguments since there is less dispersion on the circle.

The first result [2] is concerned with a priori estimates and weak solutions to KdV in H^{-1}.

Theorem 1. *Let $u_0 \in L^2$ and u the solution to KdV with initial data u_0. There exists c so that*

$$\|u(t)\|_{H^{-1}} \le c\|u_0\|_{H^{-1}}.$$

Moreover, given initial data $u_0 \in H^{-1}$ there exists a weak solution in a suitable function space which embeds into $L^\infty_{loc}(H^{-1})$.

The proof relies on the Miura map

$$v = v_x + v^2.$$

It has the following properties

(1) If v satisfies mKdV then $u = v_x + v^2$ satisfies the KdV equation.
(2) Inverting the Miura map gains roughly one derivative.

(3) There is the factorization of the Schrödinger operator

$$-\partial_{xx}^2\psi + u\psi = -(\partial_x + v)(\partial_x - v)\psi$$

if $u = v_x + v^2$. Since

$$\int (-\partial_{xx}^2\psi + u\psi)\psi dx = \|\partial_x\psi - v\psi\|_{L^2}^2$$

we see that the range of the Miura map consists of potentials corresponding to non-negative Schrödinger operators. In particular it is not surjective from $H^s \to H^{s-1}$ for $s \geq 0$. Nevertheless, for initial data in the range of the Miura map Perry, Kappeler Shubin and Topalov [12] have shown global estimates of solutions in H^{-1}. They have also shown that, if u is such a potential (lets say in $L^\infty + H^{-1}$) then there is $v \in L^2_{loc}$ which is mapped to u.

(4) Interchanging the factors is equivalent to applying the Miura map to the negative of the argument.

(5) The Miura map maps $\lambda \tanh(\lambda x)$ to λ^2 and $-\lambda \tanh(\lambda x)$ to $\lambda^2 - \lambda^2 \operatorname{sech}^2(\lambda x)$.

We adopt the following strategy: The KdV equation is invariant under the Galilean transform, i.e. if u satisfies the KdV equation then the same is true for

$$u(x - 6ht, t) + h$$

We add a constant λ^2 to the initial data to obtain a potential with positive definite Schrodinger operator. Then we show that there is a unique function $v + \lambda \tanh(\lambda(x))$, $v \in L^2$ which is mapped to $u + \lambda^2$ by the Miura map. Next we study the solution v to the modified KdV equation with this initial data. Finally we set

$$u(t - 6\lambda^2 t, x) = w_x + w^2 - \lambda^2$$

which satisfies KdV. Bounds on v turn into bounds for u.

We also obtain also as an extension of [2] a result on asymptotic stability of the soliton $u(t, x) = Q(x - 4t)$ with $Q = 2\operatorname{sech}^2(x)$.

Theorem 2. *There exists $\delta > 0$ such that the following is true: If $u \in L^2$,*

$$\|u(0) - Q\|_{H^{-1}} \leq \delta \tag{5}$$

then

$$\sup_t \inf_y \|u(t) - Q(x - y)\|_{H^{-1}} \lesssim \delta.$$

Moreover for $\varepsilon > 0$ there exists $\delta > 0$ so that the following is true: If (5) holds then there exists λ close to 1 so that

$$\inf_y \|(1 + \tanh(x_0 + x - \varepsilon t))^{1/2}(u - \lambda^2 Q(\lambda x - y))\|_{H^{-1}} \to 0 \tag{6}$$

as $t \to \infty$.

The truncation at εt cannot be avoided due to the existence of small solitons. Similar asymptotic stability in H^1 has been shown by Martel and Merle [16], and in L^2 by Merle and Vega [17].

We turn to related questions for the nonlinear Schrödinger equation. It is natural to ask whether local well-posedness also holds in negative Sobolev spaces between $H^{-1/2}$ and L^2. Uniform continuity cannot hold in this range. However, it is not implausible that one may have well-posedness with only continuous dependence on the initial data. Relaxing the exponential bound in (4) to a polynomial bound is a related challenging problem. It is not clear whether the problem differs in the focusing or the defocusing problem.

The problem of obtaining a priori estimates in negative Sobolev spaces was previously considered by Christ-Colliander-Tao [3] ($s \geq -1/12$) and by Koch and Tataru [15]($s \geq -1/6$) and [14] ($s \geq -1/4$). Key ideas are: 1) bootstrap suitable Strichartz type norms of the solution but only on frequency dependent time-scales. 2) use the I-method to construct better almost conserved H^s type norms for the problem 3) use of related local energy bounds.

In the process of proving a priori bounds , we also establish certain space-time bounds for the solution, as well as for the nonlinearity in the equation; these bounds insure that the equation is satisfied in the sense of distributions even for weak limits, and hence we also obtain existence of global weak solutions for initial data in H^s for $-1/4 \leq s < 0$. It is likely that $-1/4$ is not optimal.

The main result of [14] is as follows:

Theorem 3. *There exists $\varepsilon > 0$ such that the following is true. Let*

$$-\frac{1}{4} \leq s < 0, \qquad \Lambda \geq 1$$

and assume that the initial data $u_0 \in L^2$ satisfies

$$\|u_0\|^2_{H^s_\Lambda} := \int (\Lambda^2 + \xi^2)^s |\hat{u}_0|^2 d\xi < \varepsilon^2.$$

Then the solution u to (3) satisfies

$$\sup_{0 \leq t \leq 1} \|u(t)\|_{H^s_\Lambda} \leq 2\|u_0\|_{H^s_\Lambda}. \tag{7}$$

The above theorem captures most of the technical contents of the analysis. However, it is not scale invariant, so taking scaling into account we obtain further bounds. Indeed, rescaling

$$u_\mu(x,t) = \mu u(\mu x, \mu^2 t)$$

we have

$$\|u_\mu(0)\|_{H^s_{\mu\Lambda}} = \mu^{\frac{2s+1}{2}} \|u(0)\|_{H^s_\Lambda}. \tag{8}$$

Applying the above theorem to u_μ for $s = \frac{1}{4}$ we obtain the case $s = \frac{1}{4}$ of the following

Corollary 1.1. *Let $-\frac{1}{4} \leq s \leq 0$. Suppose that $M > 0$ and $\Lambda > 0$ satisfy $\Lambda \gg M^4$. Let u be a solution to (3) with initial data $u_0 \in L^2$ so that*

$$\|u_0\|_{H_\Lambda^{-\frac{1}{4}}} \leq M$$

Then u satisfies

$$\sup_{|t| \leq T} \|u(t)\|_{H^s_\Lambda} \lesssim \|u_0\|_{H^s_\Lambda}, \qquad \textit{for} \qquad T \ll M^{-8}. \tag{9}$$

The general case follows from the $s = \frac{1}{4}$ case due to the following equivalence:

$$\|v\|^2_{H^s_\Lambda} \approx \sum_{\lambda \geq \Lambda} \lambda^{\frac{1}{2}+2s} \|v\|^2_{H_\lambda^{-\frac{1}{4}}}. \tag{10}$$

Here and below all the λ summations are dyadic.

Applying the above corollary to a given solution for increasing values of Λ yields global in time bounds. Consider first the case when $1/4 < s < 0$. Given $M \geq 1$ and initial data u_0 so that $\|u_0\|_{H^s} \leq M$ we have

$$\|u_0\|_{H_\Lambda^{-\frac{1}{4}}} \lesssim \Lambda^{-s-\frac{1}{4}} M, \qquad \Lambda \geq 1$$

By the above corollary this yields

$$\sup_{0\leq t\leq T} \|u(t)\|_{H_\Lambda^s} \lesssim \|u_0\|_{H_\Lambda^s}, \qquad \Lambda \gg \max\{T^{\frac{1}{8s+2}} M^{\frac{4}{4s+1}}, M^{\frac{2}{2s+1}}\}$$

Hence we have proved

Corollary 1.2. *Let $-\frac{1}{4} < s < 0$ and $M \geq 1$. Let u be a solution to* (3) *with initial data $u_0 \in L^2$ so that*

$$\|u_0\|_{H^s} \leq M$$

Then for all $T > 0$ the function u satisfies

$$\sup_{|t|\leq T} \|u(t)\|_{H^s_{\Lambda(T)}} \lesssim M, \qquad \Lambda(T) = \max\{T^{\frac{1}{8s+2}} M^{\frac{4}{4s+1}}, M^{\frac{2}{2s+1}}\}. \tag{11}$$

It is unlikely that the $s = -\frac{1}{4}$ result is sharp. The case $s = -\frac{1}{4}$ is more delicate. There instead of (8) we only have

$$\lim_{\Lambda\to\infty} \|u_0\|_{H_\Lambda^{-\frac{1}{4}}} = 0 \quad \text{ for } u_0 \in H^{-\frac{1}{4}}.$$

Thus we obtain

Corollary 1.3. *Let u be a solution to* (3) *with initial data $u_0 \in L^2$. Then for all $T > 0$ the function u satisfies*

$$\sup_{|t|\leq T} \|u(t)\|_{H^{-\frac{1}{4}}_{\Lambda(T)}} \leq 1 \tag{12}$$

for some increasing function $\Lambda(T)$ which only depends on the $H^{-\frac{1}{4}}$ frequency envelope of u_0.

The a priori estimates suffice to construct global weak solutions. Using the uniform bounds (7) one may prove the following statement.

Theorem 4. *Suppose that $u_0 \in H^s$, $s \geq -\frac{1}{4}$. Then there exists a weak solution $u \in C(\mathbb{R}, H^s)$, so that for all $T > 0$ we have*

$$\sup_{-T\leq t\leq T} \|u(t)\|_{H_\Lambda^s} \leq C \tag{13}$$

with Λ depending on T and on the $H^{-\frac{1}{4}}$ frequency envelope of u_0.

2. KdV and mKdV

2.1. **The Miura map.** As indicated above the Miura map on $\pm\lambda\tanh(\lambda x) + L^2$ plays a crucial role. We study the two maps. Given $\lambda > 0$ we define

$$F_-^\lambda : L^2 \ni v \to \left(v_x + (v + 2\lambda\tanh(\lambda x))v, \int v \operatorname{sech}^2(x)\right) \in L^2 \times \mathbb{R}.$$

Proposition 2.1. *The map F_-^λ is an analytic diffeomorphism from L^2 to the open set of potentials in H^{-1} with ground state energy larger than $-\lambda^2$.*

In particular, if for the potential u such that $u+\lambda^2$ is a potential for which the Schrödinger operator is positive definite than u is in the range of the Miura map. Moreover the Miura map is injective on $\lambda\tanh(\lambda x)+L^2$.

Similarly we define

$$F_+ : L^2\times(0,\infty)\ni(v,\lambda)\to -v_x+(v-2\lambda\tanh(\lambda x))v\in H^{-1}$$

and recall that

$$F_+(0,\lambda) = -2\lambda^2\operatorname{sech}^2(\lambda x).$$

Proposition 2.2. *The map F_+ maps L^2 to the open subset of potentials on H^{-1} with at least one eigenvalue. It is an analytic diffeomorphism to its image. If $v\in L^2$ then $F(v,\lambda)$ has ground state energy $-\lambda^2$.*

2.2. **mKdV near the kink.** The (defocusing) modified KdV equation

$$v_t - v_{xxx} + 6|v|^2 v_x = 0$$

has no solitons but kinks:

$$\lambda\tanh(\lambda x+2\lambda^3 t)$$

satisfies $mKdV$. We will study solutions of the form

$$v=\tanh(x-y(t))+w$$

which satisfy

$$w_t - w_{xxx} + 2(w^3+3\tanh(x-y(t))w^2+3\tanh^2(x-y(t)w)_x = (\dot y-2)\operatorname{sech}^2(x-y(t))$$

where $\dot w$ is determined by a suitable orthogonality condition $\int w\eta(x-y(t))dx=0$ with a function η which we will choose below.

It is instructive to consider the linearized equation, in a frame moving with the kink

$$w_t - w_{xxx} + 4w_x - 6(\operatorname{sech}^2(x)w)_x = \alpha\operatorname{sech}^2(x)$$

where α is chosen so that $\int e^x\operatorname{sech}^2(x)wdx=0$ is preserved.

Then

$$\frac{d}{dt}\int e^x w^2 dx = -B(e^{x/2}w) := -3\int w_x^2+\frac{5}{4}w^2-2\operatorname{sech}^2(x)w^2-4\operatorname{sech}^2(x)\tanh(x)w^2dx$$

We claim that there exists $\kappa>0$ so that

$$B(f)\geq\kappa\|f\|^2_{H^1} \tag{14}$$

provided $\int fe^{x/2}\operatorname{sech}^2(x)dx=0$. It should be possible to give an analytic proof for this statement. In any case it is easy to check it numerically.

Theorem 5. *Let $\gamma<6$. There exists $\delta>0$ such that, if*

$$\|v-\tanh(x)\|_{L^2}<\delta$$

then

$$\sup_t\|v(t)-\tanh(x-y(t))\|_{L^2}\lesssim\delta$$

where the continuous function $y(t)$ satisfies

$$|\dot y+2|\lesssim\delta+g$$

with

$$\|g\|_{L^1} \lesssim \delta.$$

Moreover, for all $x_0 \in \mathbb{R}$*,*

$$\int (1+\tanh(x+\gamma t-x_0))|v(t)-\tanh(x-y(t))|^2 dx \lesssim \int (1+\tanh(x-x_0))|v(0,x)-\tanh(x)|^2 dx.$$

The proof has two parts: First, with $\eta = e^{-2R} + 1 + \tanh(x - y(t) + R)$

$$\frac{d}{dt}\int \eta w^2 dx \leq -\delta \int \eta'(w^2 + w_x^2) dx$$

ensures that the L^2 norm of the deviation from the kink is uniformly bounded. Second

$$\frac{d}{dt}\int \eta(1+\tanh(x+\gamma t+x_0))w(t,x)^2 dx \lesssim \int \eta'(w^2+w_x^2)dx \int \eta(1+\tanh(x+\gamma t+x_0))w(t,x)^2 dx$$

and Gronwall's inequality together with the (integrated) first bound imply the estimate.

Asymptotic stability is an immediate consequence: The right hand side becomes small if $x_0 \to \infty$, and hence so does the left hand side.

2.3. **Completion of the proof for KdV.** Let $u_0 \in L^2$. We choose λ so that λ^2 is large depending on $\|u_0\|_{H^{-1}}$. Then there exists w_0 so that $u_0 = F_-^\lambda(w_0)$, and, since λ is large,

$$\|w_0\|_{L^2} \leq c\|u_0\|_{H^{-1}}.$$

By Theorem 5 there is a unique global solution w to the mKdV near the kink with

$$\|w(t)\|_{L^2} \lesssim \|w(0)\|_{L^2}$$

Then

$$u(x,t) = F_-^\lambda(w(t, x - y(t)))(x - 6\lambda^2 t + y(t), t)$$

is the solution to KdV and it satisfies the claimed bounds.

3. The nonlinear Schrödinger equation

The nonlinear Schrödinger equation is completely integrable. Depending on whether we look at the focusing or the defocusing problem, we expect two possible types of behavior for frequency localized data.

In the defocusing case, we expect the solutions to disperse spatially. However, in frequency there should only be a limited spreading, to a range below the dyadic scale, which depends only on the L^2 size of the data. Precisely energy estimates show that for frequency localized data with L^2 norm λ, frequency spreading occurs at most up to scale λ.

In the focusing case, the expected long time behavior is a resolution into a number of solitons (possibly infinitely many) plus a dispersive part. The situation is somewhat complicated by the fact that some of these solitons may have the same speed, and thus considerable overlapping. The inverse scattering formalism provides formulas for such solutions with many interacting solitons. Nevertheless it is instructive to consider first the case of a single soliton, which in the simplest case has the form

$$u(x,t) = e^{-it}\,\mathrm{sech}(2^{-1/2}x).$$

Rescaling we get a soliton with L^2 norm λ, namely

$$u^\lambda(x,t) = e^{-it\lambda^4}\lambda^2\,\mathrm{sech}(2^{-1/2}\lambda^2 x).$$

More soliton solutions can be obtained due to the Galilean invariance. However, our function spaces here break the Galilean invariance, so the zero speed solitons are among the worst enemies.

The above solution is constant in time, up to a phase factor. It is essentially localized to an interval of size λ^{-2} in x, and of size λ^2 in frequency. It also saturates our local energy estimates in (7) for $s = -\frac{1}{4}$, exactly when $\Lambda = \lambda^4$.

In many cases error estimates for a nonlinear semiclassical ansatz for solutions are available. An example is the initial data

$$u_0(x) = \lambda \operatorname{sech}(2^{-1/2}x)$$

where a semiclassical ansatz for an approximate solution is given by

$$u(t,x) = \lambda A(x,t) e^{-i\lambda S(t,x)}$$

where $\rho = A^2$ and $\mu = A^2 \partial_x S$ satisfy the Whitham equations

$$\rho_t + \lambda \partial_x \mu = 0 \qquad \partial_t \mu + \lambda \partial_x (\mu^2/\rho \pm \rho^2/2) = 0$$

with $+$ in defocusing and $-$ in the focusing case. The Whitham equations are hyperbolic for the defocusing case and they can be solved up to an time $T \sim \lambda^{-1}$, when singularities corresponding to caustics occur. Grenier [6] has justified this ansatz up to the time when caustics occur.

The Whitham equations are elliptic in the focusing case. Akhmanov, Khokhlov and Sukhorukov [1] realized that the implicit equation

$$\mu = -2\lambda t \rho^2 \tanh\left(\frac{\rho x - \mu \lambda t}{\rho}\right), \qquad \rho = (1 + \lambda^2 t^2 \rho^2) \operatorname{sech}^2(\rho x - \mu t) \tag{15}$$

defines a solution to the Whitham equation with the λ sech initial data. The semiclassical ansatz for small semiclassical times has been studied by Thomann [20].

The direct scattering problem has been solved by Satsuma and Yajima [19]. In particular, if λ is an integer one obtains a pure soliton solution with λ solitons with velocity 0. In this case the solution is periodic with period 2. Formula (15) seems to indicate that the solution remains concentrated in an spatial area for size $\sim \ln(1+\lambda)$. The semiclassical limit has been worked out in a number of problems, see Jin, Levermore and McLaughlin [8], Kamvissis [9], Deift and Zhou [5]. and Kamvissis, McLaughlin and Miller [10].

These examples indicate that energy may spread over a large frequency interval even if the energy is concentrated at frequencies $\lesssim 1$ initially, and there are solutions with energy distributed over a large frequency interval with velocity zero. For the proof of our main result we use localization in frequency and space. These examples provide natural limits for the localization. This is reflected in the estimates and the definition of the function spaces.

3.1. **The proof.** We begin with a dyadic Littlewood-Paley frequency decomposition of the solution u,

$$u = \sum_{\lambda \geq \Lambda} u_\lambda, \qquad u_\lambda = P_\lambda u$$

where λ takes dyadic values not smaller than Λ, and u_Λ contains all frequencies up to size Λ. Here the multipliers P_λ are standard Littlewood-Paley projectors. For each such λ we also use a spatial partition of unity on the λ^{1+4s} scale,

$$1 = \sum_{j \in \mathbb{Z}} \chi_j^\lambda(x), \qquad \chi_j^\lambda(x) = \chi(\lambda^{-1-4s}x - j) \tag{16}$$

with $\chi \in C_0^\infty(-1, 1)$. To prove the theorem we will use

(i) Two energy spaces, namely a standard energy norm

$$\|u\|^2_{l^2 L^\infty H^s_\Lambda} = \sum_{\lambda \geq \Lambda} \lambda^{2s} \|u_\lambda\|^2_{L^\infty L^2} \tag{17}$$

and a local energy norm[1] adapted to the λ^{1+4s} spatial scale,

$$\|u\|^2_{l^2 l^\infty L^2 H^{-s}_\Lambda} = \sum_{\lambda \geq \Lambda} \lambda^{-2s-2} \sup_{j \in \mathbb{Z}} \|\chi_j^\lambda \partial_x u_\lambda\|^2_{L^2} \tag{18}$$

(ii) Two Banach spaces X^s_Λ and $X^s_{\Lambda,le}$ measuring the space-time regularity of the solution u. The first one measures the dyadic parts of u on small frequency dependent timescales, and is mostly similar to the spaces introduced in [3], [15]. The second is introduced in [14] , and measures the spatially localized size of the solution on the unit time scale.

(iii) Two corresponding Banach spaces Y^s_Λ and $Y^s_{\Lambda,le}$ measuring the regularity of the nonlinear term $|u|^2 u$.

The linear part of the argument is a straightforward consequence of our definition of the spaces, and is given by

Proposition 3.1. *The following estimates hold for solutions to* (3)*:*

$$\|u\|_{X^s_\Lambda} \lesssim \|u\|_{l^2 L^\infty H^s_\Lambda} + \|(i\partial_t - \Delta)u\|_{Y^s_\Lambda} \tag{19}$$

respectively

$$\|u\|_{X^s_{\Lambda,le}} \lesssim \|u\|_{l^2 l^\infty L^2 H^{-s}_\Lambda} + \|(i\partial_t - \Delta)u\|_{Y^s_{\Lambda,le}} \tag{20}$$

To estimate the nonlinearity we need a cubic bound,

Proposition 3.2. *Let* $u \in X^s_\Lambda \cap X^s_{\Lambda,le}$. *Then* $|u|^2 u \in Y^s_\Lambda \cap Y^s_{\Lambda,le}$ *and*

$$\||u|^2 u\|_{Y^s_\Lambda \cap Y^s_{\Lambda,le}} \lesssim \|u\|^3_{X^s_\Lambda \cap X^s_{\Lambda,le}} \tag{21}$$

Finally, to close the argument we need to propagate the energy norms:

Proposition 3.3. *Let u be a solution to* (3) *with*

$$\|u\|_{l^2 L^\infty H^s_\Lambda} \ll 1.$$

Then we have the energy bound

$$\|u\|_{l^2 L^\infty H^s_\Lambda} \lesssim \|u_0\|_{H^s_\Lambda} + \|u\|^3_{X^s_\Lambda \cap X^s_{\Lambda,le}}, \tag{22}$$

[1] For $s = -\frac{1}{4}$ the spatial scale is one and this corresponds to the familiar gain of one half of a derivative. It may seem more natural to remove the ∂_x derivative and appropriately adjust the power of λ. This would be equivalent for all frequencies $\lambda > \Lambda$. However, in u_Λ we are including all lower frequencies, which correspond to waves with lower group velocities and to a worse local energy bound.

respectively the local energy decay

$$\|u\|_{l^2 l^\infty L^2 H^{-s}_\Lambda} \lesssim \|u_0\|_{H^s_\Lambda} + \|u\|^3_{X^s_\Lambda \cap X^s_{\Lambda,le}}. \tag{23}$$

A standard bootstrap argument leads from Propositions 3.1,3.2 and 3.3 to Theorem 3.

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