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## $L^2$ -STABILITY OF MULTI-SOLITONS

CLAUDIO MUÑOZ

### 1. INTRODUCTION

The aim of this note is to give a short review of our recent work (see [5]) with Miguel A. Alejo and Luis Vega, concerning the  $L^2$ -stability, and asymptotic stability, of the  $N$ -soliton of the Korteweg-de Vries (KdV) equation

$$u_t + (u_{xx} + u^2)_x = 0. \quad (1.1)$$

Here  $u = u(t, x)$  is a real valued function, and  $(t, x) \in \mathbb{R}^2$ . This equation arises in Physics as a model of propagation of dispersive long waves, as was pointed out by Russel in 1834 [30]. The exact formulation of the KdV equation comes from Korteweg and de Vries (1895) [18]. This equation was studied in a numerical work by Fermi, Pasta and Ulam, and by Kruskal and Zabusky [13, 19].

From the mathematical point of view, equation (1.1) is an *integrable model* [2, 3, 20, 1], with infinitely many conservation laws. Moreover, since the Cauchy problem associated to (1.1) is locally well posed in  $L^2(\mathbb{R})$  (cf. Bourgain [8]), each solution is indeed global in time thanks to the *Mass* conservation law

$$M[u](t) := \frac{1}{2} \int_{\mathbb{R}} u^2(t, x) dx = M[u](0). \quad (1.2)$$

On the other hand, equation (1.1) has solitary wave solutions referred as *solitons*, solutions of the form

$$u(t, x) = Q_c(x - ct), \quad Q_c(s) := cQ(\sqrt{cs}), \quad c > 0, \quad (1.3)$$

and

$$Q(s) := \frac{3}{1 + \cosh(s)}. \quad (1.4)$$

The study of perturbations of solitons or solitary waves lead to the introduction of the concepts of *orbital and asymptotic stability*. In particular, it is natural to expect that solitons are stable in the energy space  $H^1(\mathbb{R})$ . Indeed,  $H^1$ -stability of KdV solitons has been considered by Benjamin and Bona-Souganidis-Strauss in [6, 7]. On the other hand, the asymptotic stability has been studied e.g. in Pego-Weinstein and Martel-Merle [35, 22].

Concerning the more involved case of the sum of  $N \geq 2$  decoupled solitons, stability and asymptotic stability results are very recent. First of all, recall that, as

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a consequence of the integrability property, KdV allows the existence of solutions behaving, as time goes to infinity, as the sum of  $N$  decoupled solitons. These solutions are well-known in the literature and are called  $N$ -solitons, or generically *multi-solitons* [14]. Indeed, any  $N$ -soliton solution has the form  $u(t, x) := U^{(N)}(t, x) := U^{(N)}(x; c_j, x_j - c_j t)$ , where

$$\{U^{(N)}(x; c_j, y_j) : c_j > 0, y_j \in \mathbb{R}, j = 1, \dots, N\} \quad (1.5)$$

is a family of explicit  $N$ -soliton profiles (see e.g. Maddocks-Sachs [21], §3.1). In particular, this solution describes multiple soliton collisions; but since solitons for KdV equation interact in a linear fashion, there is no residual appearing after the collisions, even if the equation is nonlinear in nature. In other words,

$$\lim_{t \rightarrow \pm\infty} \|U^{(N)}(t) - \sum_{j=1}^N Q_{c_j}(\cdot - c_j t - x_j^\pm)\|_{H^1(\mathbb{R})} = 0,$$

with  $x_j^\pm \in \mathbb{R}$  depending on the set  $(c_k)$ . This is also a consequence of the integrability property.

In [21], Maddocks and Sachs considered the  $H^N(\mathbb{R})$ -stability of the  $N$ -soliton solution of KdV, by using  $N + 1$  conservation laws. Next, in [27, 26], Martel, Merle and Tsai improved the preceding result by proving stability and asymptotic stability of the *sum of  $N$  solitons, well decoupled* at the initial time, in the energy space. Their proof also applies for general nonlinearities and not only for the integrable cases, provided they have stable solitons, in the sense of Weinstein [37]. Their approach is based on the construction of  $N$  *almost conserved quantities*, related to the mass of each solitary wave, and the total energy of the solution. As a consequence of the existence of  $N$ -soliton solutions for KdV, the above results can be extended to give a global stability property, improving the Maddock-Sachs results. See also [23, 24, 25] for global  $H^1$ -stability results in some non-integrable cases.

As far as we know, the unique stability result for KdV solitons, below  $H^1(\mathbb{R})$ , was proved by Merle and Vega in [28]. Precisely, in that work, the authors prove that solitons of (1.1) are  $L^2$ -stable, by using the *Miura transform*

$$M[v] := \frac{3}{\sqrt{2}}v_x - \frac{3}{2}v^2, \quad (1.6)$$

which links solutions of the *defocusing, modified KdV* equation,

$$v_t + (v_{xx} - v^3)_x = 0, \quad v = v(t, x) \in \mathbb{R}, \quad (t, x) \in \mathbb{R}^2, \quad (1.7)$$

with solutions of the KdV equation (1.1). In particular, the image of the family of *kink* solutions  $\varphi_c(t, x) := \sqrt{c} \tanh(\sqrt{c/2}(x + ct + x_0))$ ,  $c > 0$ ,  $x_0 \in \mathbb{R}$ , of (1.7) under the transformation (1.6) is the soliton  $Q_c$  above described, up to a standard Galilean transformation (cf. [28]).

Let us describe in more detail the Merle-Vega's approach. First of all, from the fact that the image of mKdV kinks are KdV solitons, and using the fact that the soliton is a minimizer of a well-known  $H^1$  functional, one can define the inverse of the Miura transform (1.6) in a small  $L^2$ -vicinity of the soliton  $Q_c$ , to obtain a small  $H^1$  neighborhood of the kink solution. Since the kink solution of (1.7) is  $H^1$ -stable (see e.g. Zhidkov, Merle-Vega [39, 28]), by applying once again the Miura transform to the mKdV solution, and using a well-known unicity argument,

the authors concluded the  $L^2$ -stability of the KdV soliton. The following figure describes the aforementioned approach:

$$\begin{array}{ccc}
 \text{KdV} & \xrightarrow[\text{Miura}]{v_0 = M^{-1}[u_0]} & \text{mKdV} \\
 u_0 \sim_{L^2} Q_c & & v_0 \sim_{H^1} \text{kink} \\
 \\ 
 L^2\text{-KdV flow} & & H^1\text{-mKdV flow} \\
 \text{(Bourgain)} \downarrow t > 0 & & \text{(Merle-Vega)} \downarrow H^1\text{-stability} \\
 & & \text{(Zhidkov)} \\
 \\ 
 u(t) = \bar{u}(t) & \xleftarrow[\text{Miura}]{\bar{u}(t) = M[v](t)} & v(t) \text{ stable} \\
 u(0) = u_0 & & v(0) = v_0
 \end{array} \tag{1.8}$$

Fig. 1: *The Merle-Vega's approach.*

The Merle-Vega's idea has been applied to different models describing several phenomena. A similar Miura transform is available for the KP II equation, a two-dimensional generalization of the KdV equation. Mizumachi and Tzvetkov have shown the stability of solitary waves of KdV, seen as solutions of KP II, under periodic transversal perturbations [32]. Finally, we recall the  $L^2$ -stability result for solitary waves of the cubic NLS equation proved by Mizumachi and Pelinovsky in [31]. Other applications of the Miura transform are local well and ill-posedness results (cf. [17, 10]).

A natural question to consider is the generalization of the Merle-Vega's result to the case of multi-soliton solutions. In [12] (see also [36]), the authors state that the Miura transform sends *multi-kink* solutions of (1.7) towards a well defined family of *multi-soliton* solutions of (1.1). However, we have found that multi-kinks are hard to manipulate, due to the continuous interaction of non-local terms (recall that a kink does not belong to  $L^2(\mathbb{R})$ ).<sup>1</sup> Therefore we have followed a different approach.

Indeed, in [5] we considered a *Gardner transform* [29, 11], well-known in the mathematical and physical literature since the late sixties, and which links  $H^1$ -solutions of the Gardner equation

$$v_t + (v_{xx} + v^2 - \beta v^3)_x = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x, \quad \beta > 0, \tag{1.9}$$

with  $L^2$ -solutions of the KdV equation (1.1). More specifically, given any  $\beta > 0$  and  $v(t) \in H^1(\mathbb{R})$ , solution of the Gardner equation (1.9), the Gardner transform [11]

$$u(t) = M_\beta[v](t) := \left[ v - \frac{3}{2}\sqrt{2\beta}v_x - \frac{3}{2}\beta v^2 \right](t), \tag{1.10}$$

is an  $L^2$ -solution of KdV. Compared with the original Miura transform (1.6), it has an additional *linear* term which simplifies the proofs.

In addition, the Gardner equation is also an integrable model [11], with soliton solutions of the form

$$v(t, x) := Q_{c,\beta}(x - ct),$$

<sup>1</sup>See [34] for a recent proof of the stability of multi-kinks of mKdV.

and<sup>2</sup>

$$Q_{c,\beta}(s) := \frac{3c}{1 + \rho \cosh(\sqrt{c}s)}, \quad \text{with} \quad \rho := \left(1 - \frac{9}{2}\beta c\right)^{1/2}, \quad 0 < c < \frac{2}{9\beta}. \quad (1.11)$$

In particular, in the formal limit  $\beta \rightarrow 0$ , we recover the standard KdV soliton (1.3)-(1.4). On the other hand, the Cauchy problem associated to (1.9) is globally well-posed under initial data in the energy class  $H^1(\mathbb{R})$  (cf. Kenig-Ponce-Vega [16]), thanks to the mass (1.2) and *energy* conservation laws.

We have been interested in the image of the family of solutions (1.11) under the aforementioned, Gardner transform. Surprisingly enough, it turns out that the resulting family is **nothing but** the KdV soliton family (1.3). Indeed, a direct computation shows that for the Gardner soliton solution (1.11), one has

$$\begin{aligned} M_\beta[Q_{c,\beta}](t) &= \left[Q_{c,\beta} - \frac{3}{2}\sqrt{2\beta}Q'_{c,\beta} - \frac{3}{2}\beta Q_{c,\beta}^2\right](x - ct) \\ &= Q_c(x - ct - \delta), \end{aligned} \quad (1.12)$$

with  $\delta = \delta(c, \beta) > 0$  provided  $\beta > 0$ , and  $Q_c$  the KdV soliton solution (1.3). In other words, the Gardner transform (1.10) sends the Gardner soliton towards a slightly translated KdV soliton. This last fact formally suggests that multi-soliton solutions of the Gardner equation (1.9) are sent towards (or close enough to) multi-soliton solutions of the KdV model (1.1), as is done in [36] for the case of the Miura transform.

In [5], we profit of this property to improve the  $H^1$ -stability and asymptotic stability properties proved by Martel, Merle and Tsai in [27], and Martel and Merle [26], now in the case of  $L^2$ -perturbations of the KdV multi-solitons. In order to maintain the presentation the simplest possible, we prefer do not state the asymptotic stability result that we have obtained. See [5] for more details.

**Theorem 1.1** ( $L^2$ -stability of the  $N$ -soliton, [5]). *Let  $\delta > 0$ ,  $N \geq 2$ ,  $0 < c_1^0 < \dots < c_N^0$  and  $x_1^0, \dots, x_N^0 \in \mathbb{R}$ . There exists  $\alpha_0 > 0$  such that if  $0 < \alpha < \alpha_0$ , then the following holds. Let  $u(t)$  be a solution of (1.1) such that*

$$\|u(0) - U^{(N)}(\cdot; c_j^0, -x_j^0)\|_{L^2(\mathbb{R})} \leq \alpha,$$

*with  $U^N$  the  $N$ -soliton profile described in (1.5). Then there exist  $x_j(t)$ ,  $j = 1, \dots, N$ , such that*

$$\sup_{t \in \mathbb{R}} \|u(t) - U^{(N)}(\cdot; c_j^0, -x_j(t))\|_{L^2(\mathbb{R})} \leq \delta. \quad (1.13)$$

The above result can be seen as a consequence of the stability of an initial datum close enough to the **sum of  $N$  decoupled solitons** of the KdV equation, and the uniform continuity of the KdV flow for  $L^2$ -data, see e.g. [27], Corollary 1.

**Theorem 1.2** ( $L^2$ -stability of the sum of  $N$  solitons of KdV). *Let  $N \geq 2$  and  $0 < c_1^0 < c_2^0 < \dots < c_N^0$ . There exist parameters  $\alpha_0, A_0, L, \gamma > 0$ , such that the following holds. Consider  $u_0 \in L^2(\mathbb{R})$ , and assume that there exist  $L > L_0$ ,  $\alpha \in (0, \alpha_0)$  and  $x_1^0 < x_2^0 < \dots < x_N^0$ , such that*

$$x_j^0 > x_{j-1}^0 + L, \quad \text{with} \quad j = 2, \dots, N,$$

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<sup>2</sup>See e.g. [9, 33] and references therein for a more detailed description of solitons and integrability for the Gardner equation.

and

$$\|u_0 - R_0\|_{L^2(\mathbb{R})} \leq \alpha, \quad \text{with} \quad R_0 := \sum_{j=1}^N Q_{c_j^0}(\cdot - x_j^0). \quad (1.14)$$

Then there exist  $x_1(t), \dots, x_N(t)$  such that the solution  $u(t)$  of the Cauchy problem for the KdV equation (1.1), with initial data  $u_0$ , satisfies

$$\sup_{t \geq 0} \left\| u(t) - \sum_{j=1}^N Q_{c_j^0}(\cdot - x_j(t)) \right\|_{L^2(\mathbb{R})} \leq A_0(\alpha + e^{-\gamma_0 L}). \quad (1.15)$$

In the next section we sketch the proof of this last result. As a final remark, we believe that using the methods developed in [5], the stability –under periodic transversal perturbations– of the KdV multi-soliton  $U^{(N)}$ , seen as a solution of the KP II equation constant in the  $y$ -variable, can be handled via a Gardner transform, improving the results by Mizumachi and Tzvetkov [32].

## 2. PROOF OF THEOREM 1.2

Let  $u_0 \in L^2(\mathbb{R})$  satisfying (1.14). Let us denote by  $z_0 := u_0 - R_0$ , such that  $\|z_0\|_{L^2(\mathbb{R})} \leq \alpha$ . We want to solve the nonlinear Riccati equation

$$M_\beta[v_0] = u_0 = R_0 + z_0, \quad (2.1)$$

with  $M_\beta$  the Gardner transform given by (1.10). We will prove that provided  $\alpha$  and  $\beta$  are small enough, and using a fixed point argument, instead of the original Merle-Vega's idea.

**Proposition 2.1** (Local invertibility around  $R_0$ ). *There exists  $\beta_0 > 0$  such that, for all  $0 < \beta < \beta_0$ , the following holds. There exist  $K_0, L_0, \gamma_0, \alpha_0 > 0$  such that for all  $0 < \alpha < \alpha_0$ ,  $L > L_0$ , and  $\|z_0\|_{L^2(\mathbb{R})} \leq \alpha$ , there exists a solution  $v_0 \in H^1(\mathbb{R})$  of (2.1), such that*

$$\|v_0 - S_0\|_{H^1(\mathbb{R})} \leq K_0 \left( \frac{\alpha}{\sqrt{\beta}} + e^{-\gamma_0 L} \right), \quad (2.2)$$

with  $S_0(x) := \sum_{j=1}^N Q_{c_j^0, \beta}(x - x_j^0 - \delta_j)$ ,

$$\delta_j = \delta_j(c_j^0) := (c_j^0)^{-1/2} \cosh^{-1} \left( \frac{1}{\rho_j} \right),^3 \quad \rho_j := \left( 1 - \frac{9}{2} \beta c_j^0 \right)^{1/2}, \quad j = 1, \dots, N, \quad (2.3)$$

and  $Q_{c, \beta}$  being the soliton solution of the Gardner equation (1.9).

Let us sketch the proof of this result. Assume  $\beta > 0$  small, such that  $\delta_j = O(\beta)$ , independent of  $c_j^0$ . Note also that  $S_0 \in H^1(\mathbb{R})$  with  $\|S_0\|_{H^1(\mathbb{R})} \leq K$ , independent of  $\beta$ . Moreover, a direct computation, using (1.12) shows that

$$M_\beta[S_0](t) = R_0 + O_{L^2(\mathbb{R})}(\beta e^{-\gamma_0 L}), \quad (2.4)$$

for some  $\gamma_0 > 0$ , independent of  $\beta$  small. Then, we look for a solution  $v_0 \in H^1(\mathbb{R})$  of (2.1), of the form  $v_0 = S_0 + w_0$ , and  $w_0$  small in  $H^1(\mathbb{R})$ . In other words,  $w_0$  has to solve the nonlinear equation

$$\mathcal{L}[w_0] = (R_0 - M_\beta[S_0]) + z_0 + \frac{3}{2} \beta w_0^2, \quad (2.5)$$

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<sup>3</sup>We take the positive inverse.

with

$$\mathcal{L}[w_0] := -\frac{3}{2}\sqrt{2\beta}w_{0,x} + (1 - 3\beta S_0)w_0. \quad (2.6)$$

We can see  $\mathcal{L}$  as a unbounded operator in  $L^2(\mathbb{R})$ , with dense domain  $H^1(\mathbb{R})$ . From standard energy estimates (see [5]), one has that for  $\beta > 0$  small enough, any solution  $w_0 \in H^1(\mathbb{R})$  of the linear problem

$$\mathcal{L}[w_0] = f, \quad f \in L^2(\mathbb{R}), \quad (2.7)$$

must satisfy, for some fixed constant  $K_0 > 0$ ,

$$\|w_0\|_{H^1(\mathbb{R})} \leq \frac{K_0}{\sqrt{\beta}} \|f\|_{L^2(\mathbb{R})}. \quad (2.8)$$

In order to prove the existence and uniqueness of a solution of (2.7), we use (2.8) and a fixed point approach, in the spirit of [38, 15]. See [5] for the details.

In what follows, let us denote by  $T := \mathcal{L}^{-1} : L^2(\mathbb{R}) \rightarrow H^1(\mathbb{R})$  the resolvent operator above mentioned. Now, from (2.5), we want to solve the nonlinear problem

$$w_0 = T[w_0] = \mathcal{L}^{-1}[(R_0 - M_\beta[S_0]) + z_0 + \frac{3}{2}\beta w_0^2]. \quad (2.9)$$

In order to invoke, once again, a fixed point argument, we consider the ball

$$\mathcal{B} := \left\{ w_0 \in H^1(\mathbb{R}) \mid \|w_0\|_{H^1(\mathbb{R})} \leq 2K_0 \left( \frac{\alpha}{\sqrt{\beta}} + e^{-\gamma_0 L} \right) \right\},$$

with  $K_0 > 0$  the constant from (2.8), and  $\gamma_0 > 0$  given in (2.4). A direct argument shows that  $T$  is a contraction mapping from  $\mathcal{B}$  into itself, provided  $\beta$  is small. The proof is now complete.

The above proposition allows to construct the inverse of the Gardner transform in a small  $L^2$ -vicinity of  $R_0$ . Now we follow a similar approach to that of [28], applied this time to the Gardner equation. In order to do this, recall the following stability result for Gardner solitons, proved by Martel and Merle in the general case of gKdV equations.

**Proposition 2.2** ( $H^1$ -stability for Gardner solitons, [27, 26]). *Let  $0 < c_1^0 < c_2^0 < \dots < c_N^0 < \frac{2}{9\beta}$  be such that*

$$\partial_c \int_{\mathbb{R}} Q_{c,\beta}^2 \Big|_{c=c_j} > 0, \quad \text{for all } j = 1, \dots, N. \quad (2.10)$$

*There exists  $\tilde{\alpha}_0, \tilde{A}_0, \tilde{L}_0, \tilde{\gamma} > 0$  such that the following is true. Let  $v_0 \in H^1(\mathbb{R})$ , and assume that there exists  $\tilde{L} > \tilde{L}_0$ ,  $\tilde{\alpha} \in (0, \tilde{\alpha}_0)$  and  $\tilde{x}_1^0 < \tilde{x}_2^0 < \dots < \tilde{x}_N^0$ , such that*

$$\|v_0 - \sum_{j=1}^N Q_{c_j^0, \beta}(\cdot - \tilde{x}_j^0)\|_{H^1(\mathbb{R})} \leq \tilde{\alpha}, \quad (2.11)$$

$$\tilde{x}_j^0 > \tilde{x}_{j-1}^0 + \tilde{L}, \quad j = 2, \dots, N. \quad (2.12)$$

*Then there exists  $\tilde{x}_1(t), \dots, \tilde{x}_N(t)$  such that the solution  $v(t)$  of the Cauchy problem associated to (1.9), with initial data  $v_0$ , satisfies*

$$v(t) = S(t) + w(t), \quad S(t) := \sum_{j=1}^N Q_{c_j^0, \beta}(\cdot - \tilde{x}_j(t)),$$

and

$$\sup_{t \geq 0} \left\{ \|w(t)\|_{H^1(\mathbb{R})} + \sum_{j=1}^N |\tilde{x}'_j(t) - c_j| \right\} \leq \tilde{A}_0(\tilde{\alpha} + e^{-\tilde{\gamma}\tilde{L}}). \quad (2.13)$$

We apply this property to the initial datum  $v_0$  obtained in Proposition 2.1. After this point, the proof of Theorem 1.2 follows closely the ideas of [28], giving the desired result. We finish with the following diagram, which describes the approach we have followed.

$$\begin{array}{ccc} \text{KdV} & \xrightarrow{\text{Gardner}} & \text{Gardner} \\ u_0 \sim_{L^2} R_0 & \xrightarrow{v_0 = M_\beta^{-1}[u_0]} & v_0 \sim_{H^1} S_0 \end{array}$$
  

$$\begin{array}{ccc} L^2\text{-KdV flow} & & H^1\text{-Gardner flow} \\ \text{(Bourgain)} \downarrow & t > 0 & \downarrow \text{H}^1\text{-stability} \\ & & \text{(K-P-V)} \quad \downarrow \text{(Martel-Merle)} \end{array}$$
  

$$u(t) = \bar{u}(t) \xleftarrow[\bar{u}(t) = M_\beta[v](t)]{\text{Gardner}} v(t) \text{ stable}$$

Fig. 2: *The Gardner's approach.*

See [5] for a detailed proof.

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