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A consistence-stability approach to hydrodynamic limit of interacting particle systems on lattices


https://doi.org/10.5802/slsedp.154


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A CONSISTENCE-STABILITY APPROACH TO HYDRODYNAMIC LIMIT OF INTERACTING PARTICLE SYSTEMS ON LATTICES

ANGELIKI MENEGAKI AND CLÉMENT MOUHOT

Abstract. This is a review based on the presentation done at the seminar Laurent Schwartz in December 2021. It is announcing results in the forthcoming [MMM22]. This work presents a new simple quantitative method for proving the hydrodynamic limit of a class of interacting particle systems on lattices. We present here this method in a simplified setting, for the zero-range process and the Ginzburg-Landau process with Kawasaki dynamics, in the parabolic scaling and in dimension 1. The rate of convergence is quantitative and uniform in time. The proof relies on a consistence-stability approach in Wasserstein distance, and it avoids the use of both the so-called “block estimates”.

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1. The general method

We consider the hydrodynamic limit of interacting particle systems on a lattice. The problem is to show that under an appropriate scaling of time and space, the local particle densities of a stochastic lattice gas converge to the solution of a macroscopic partial differential equation. We first present our method abstractly and then sketch applications to three concrete models: the simple-exclusion process (SEP), the zero-range process (ZRP) and the Ginzburg Landau process with Kawasaki dynamics (GLK). The hydrodynamic limit is known at a qualitative level for all these models under both hyperbolic and parabolic scalings for the SEP and ZRP and under parabolic scaling for the GLK, see [GPV88, Yau91, Rez91, KL99]. However finding quantitative error estimates had remained an important opened question, as well as understanding the long-time behaviour of the hydrodynamic limit. First results towards quantitative error, in the particular case of the Ginzburg-Landau process with Kawasaki dynamics in dimension 1, were obtained in the two-parts work [DMOWa, DMOWb], which builds upon partial progresses in [GOVW09].

1.1. Set up and notation. We denote by $X$ the state space at a given site (number of particles, spin, etc.), which will in this paper be $\mathbb{N}$ (ZRP) or $\mathbb{R}$ (GLK). Consider the discrete torus $T_N^d$ and the corresponding phase space of particle configurations $X_N := X^{\mathbb{Z}^d}_N$. Variables in $T_N^d$ are called microscopic and denoted by $x, y, z$, whereas variables in the limit continuous torus $T^d$ are called macroscopic and denoted by $u$;

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finally particle configurations in $X_N$ are denoted by $\eta$. The canonical embedding $T^d_N \to T^d$, $\eta \mapsto \eta/N \in M_+(T^d)$ means the macroscopic distance between sites of the lattice is $1/N$. Given a particle configuration $\eta \in X_N$, we define the empirical measure

$$\alpha^N_\eta := \sum_{x \in T^d_N} \eta_x \delta_{x/N} \in M_+(T^d).$$

where $\eta_x$ denotes the value of $\eta$ at $x \in T^d_N$, and $M_+(T^d)$ is the space of positive Radon measures on the torus, and $\sum$ denotes the “average sum”, here $N^{-d} \sum_{x \in T^d_N}$.

At the microscopic level, the interacting particle system evolves through a stochastic process and the time-dependent probability measure describing the law of $\eta$ is denoted by $\mu^N_t \in P(X_N)$. We consider a linear operator $\mathcal{L}_N : C_b(X_N) \to C_b(X_N)$ generating uniquely a Feller semigroup $e^{t\mathcal{L}_N}$ on $P(X_N)$ (see [Lig85, Chapter 1]) so that given $\mu_0^N \in P(X_N)$ the solution $\mu^N_t = e^{t\mathcal{L}_N} \mu_0^N \in P(X_N)$ satisfies

$$\forall \Phi \in C_b(X_N), \quad \frac{d}{dt} \langle \Phi, \mu^N_t \rangle = \langle \mathcal{L}_N \Phi, \mu^N_t \rangle,$$

where $C_b(X_N)$ denotes continuous bounded real-valued functions and $\langle \cdot, \cdot \rangle$ denotes the duality bracket between $C_b(X_N)$ and $P(X_N)$.

At the macroscopic level, we consider a map $\mathcal{L}_\infty : \mathcal{M}_+(G^\infty) \to \mathcal{M}_+(G^\infty)$ (in general unbounded and nonlinear) and the evolution problem

$$\partial_t f_t = \mathcal{L}_\infty f_t, \quad f_t = f_0.$$

A measure $\mu^N \in P(X_N)$ is called invariant for (1.2) if

$$\forall \Phi \in C_b(X_N), \quad \langle \mu^N, \mathcal{L}_\infty \Phi \rangle = 0.$$

We also denote $\text{Lip}(X_N)$ the Lipschitz functions $\Phi : X_N \to \mathbb{R}$ with respect to the (normalised) $\ell^1$ norm: for every $\eta, \zeta \in X_N$, $|\Phi(\eta) - \Phi(\zeta)| \leq C_{\Phi} \sum_{x \in T^d_N} |\eta_x - \zeta_x|$, and we denote the smallest such constant $C_{\Phi}$ as $[\Phi]_{\text{lip}(X_N)} \in \mathbb{R}^+$. We then define, given a macroscopic profile $f$ on $T^d$, the local Gibbs measure

$$\vartheta^N_\eta (\eta) := \nu^N_{\sigma(f(x))} (\eta) \quad \text{where} \quad \nu^N_f (\eta) := \prod_{x \in \Sigma_N} n_{\sigma_f(x)} (\eta(x)).$$

The two maps $\eta \mapsto \alpha^N_\eta$ and $f \mapsto \vartheta^N_\eta$ allow comparisons between the microscopic and macroscopic scales, as summarized in Figure 1.

**Figure 1.** The functional setting.
(H1) **Microscopic stability.** The semigroup $e^{t\mathcal{L}_N}$ satisfies
\begin{equation}
\forall \Phi \in \text{Lip}(X_N), \quad [e^{t\mathcal{L}_N}\Phi]_{\text{Lip}(X_N)} \leq [\Phi]_{\text{Lip}(X_N)}. \tag{1.4}
\end{equation}

(H2) **Macroscopic stability.** There is a Banach space $\mathfrak{B} \subset \mathcal{M}_+(G_\infty)$ so that (1.3) is locally well-posed in $\mathfrak{B}$; given the maximal time of existence $T_m \in [0, +\infty]$ we denote for $t \in [0, T_m)$, $R(t) := \|f_t - f_\infty\|_{\mathfrak{B}}$ when (1.3) has a unique stationary solution $f_\infty \in \mathfrak{B}$ with mass $\int_{\mathbb{T}^d} f_\infty = \int_{\mathbb{T}^d} f_0$, otherwise we denote $R(t) := \|f_t\|_{\mathfrak{B}}$.

(H3) **Consistency.** There is a consistency error $\epsilon(N) \to 0$ as $N \to \infty$ so that for $T \in [0, T_m)$
\[\frac{1}{T} \int_0^T \int_0^t \left( e^{(t-s)\mathcal{L}_N} \Phi \right) \left[ \mathcal{L}_N^* \left( \frac{d\theta_N^N}{d\nu_\infty} \right) - \frac{d}{ds} \left( \frac{d\theta_N^N}{d\nu_\infty} \right) \right] \, ds \, dt \leq \epsilon(N) [\Phi]_{\text{Lip}(X_N)} \int_0^T R(s) \, ds\]
for any $\Phi \in \text{Lip}(X_N)$, where $\nu_\infty^N$ is an equilibrium measure.

1.3. The abstract strategy.

**Theorem 1.1.** Consider (1.2)-(1.3) with the assumptions (H0)–(H1)–(H2)–(H3). Let $\phi \in C^\infty(\mathbb{T}^d)$, $\mu_0^N \in \mathcal{P}_1(X_N)$ for all $N \geq 1$, $f_0 \in \mathfrak{B}$. Then
\begin{equation}
\forall T \in [0, T_m), \quad \frac{1}{T} \int_0^T \|\mu_t^N - \theta_t^N\|_{\text{Lip}^*} \, dt \leq \epsilon(N) \int_0^T R(s) \, ds + \|\mu_0^N - \theta_0^N\|_{\text{Lip}^*}. \tag{1.5}
\end{equation}

**Remark 1.** Note that $\|\mu_t^N - \theta_t^N\|_{\text{Lip}^*} \to 0$ as $N \to \infty$ implies that the empirical measure (1.1) sampled from the law $\mu_t^N$ satisfies
\begin{equation}
\forall \phi \in C_b(G), \forall \epsilon > 0, \forall t \geq 0, \lim_{N \to \infty} \mu_t^N \left( \{ |\langle \phi_{N}, \varphi \rangle - \langle f_t, \varphi \rangle | > \epsilon \} \right) = 0 \tag{1.6}
\end{equation}
with a rate of convergence (thus recovering quantitatively results from [GPV88]):
\[
\mu_t^N \left( \{ |\langle \phi_{N}, \varphi \rangle - \langle f_t, \varphi \rangle | > \epsilon \} \right) \\
\leq \mu_t^N \left( \{ |\langle \phi_{N}, \varphi \rangle \geq \langle f_t, \varphi \rangle + \epsilon \} \right) + \mu_t^N \left( \{ |\langle \phi_{N}, \varphi \rangle \leq \langle f_t, \varphi \rangle - \epsilon \} \right) \\
\leq \int_{X_N} \left[ F^+_t \left( (\phi, \alpha_{N}) \right) - F^+_t \left( (\phi, f_t) \right) \right] d\mu_t^N + \int_{X_N} \left[ F^-_t \left( (\phi, \alpha_{N}) \right) - F^-_t \left( (\phi, f_t) \right) \right] d\mu_t^N
\]
where $F^\pm_t$ are mollified version of the characteristic functions of respectively $\{ z \geq |\phi, f_t | + \epsilon \}$ and $\{ z \leq |\phi, f_t | - \epsilon \}$, which yields
\[
\sup_{t \in [0, T]} \mu_t^N \left( \{ |\langle \phi_{N}, \varphi \rangle - \langle f_t, \varphi \rangle | > \epsilon \} \right) \leq \epsilon^{-1} \|\mu_t^N - \theta_t^N\|_{\text{Lip}^*} + \epsilon^{-2} N^{-d}.
\]

2. Concrete applications.

We apply the abstract result to two archetypical models, the zero-range process (ZRP), and the Ginzburg-Landau process with Kawasaki dynamics (GLK).

2.1. The ZRP. In this case, the state space at each site is $X = \mathbb{N}$. Given the choice of a transition function $p \in P(\mathbb{T}^d_N)$ with $p(0) = 0$ and a jump rate function $g : \mathbb{N} \to \mathbb{R}_+$, the base generator $\mathcal{L}_N$ writes
\begin{equation}
\forall \Phi \in C_b(X_N), \forall \eta \in X_N, \quad \mathcal{L}_N \Phi(\eta) := \sum_{x, y \in \mathbb{T}^d_N} p(y - x)g(\eta_x) [\Phi(\eta^y) - \Phi(\eta)]. \tag{2.1}
\end{equation}
where $\eta^{xy}$ is defined as before. The local equilibrium structure of (H0) is given by
\begin{equation}
\tag{2.2} n_{\lambda}(k) := \frac{\lambda^k}{g(k)!Z(\lambda)} \quad \text{with} \quad Z(\lambda) := \sum_{k=0}^{\infty} \frac{\lambda^k}{g(k)!}
\end{equation}
\begin{equation}
\tag{2.3} \sigma \text{ is defined implicitly by } Z(\lambda) \sigma(\rho) = Z(\sigma(\rho)) \equiv \rho
\end{equation}
denoting $g(k)! := g(k)g(k-1)\cdots g(1)$. The pair $(g, \sigma)$ thus constructed satisfies $E_n[x] = \sigma(n)$. When $\varphi \equiv \rho \in [0, +\infty)$ is constant, the local Gibbs measure $\vartheta^\rho_N \equiv \vartheta_N$ is invariant with average number of particles $\rho$. The mean transition rate is defined by $\gamma := \sum_{x \in \mathbb{Z}^d} xp(x) \in \mathbb{R}^d$. When $\gamma \neq 0$, the first non-zero asymptotic dynamics as $N \to \infty$ is given by the hyperbolic scaling $\mathcal{L}_N := N\hat{\mathcal{L}}_N$, and the corresponding limit equation is $\partial_t f = \gamma \cdot \nabla [\sigma(f)]$. When $\gamma = 0$, the first non-zero asymptotic dynamics as $N \to \infty$ is the given by the parabolic scaling $\mathcal{L}_N := N^2\hat{\mathcal{L}}_N$, and the corresponding limit equation is formally
\begin{equation}
\tag{2.4} \partial_t f = \Delta_{\sigma(f)} \quad \text{with} \quad \Delta_{\sigma} := \sum_{i,j=1}^d a_{ij} \partial^2_{x_j}
\end{equation}
\begin{equation}
\tag{2.5} \sup_{T \geq 0} \frac{1}{T} \int_0^T \left\| \mu_t^N - \vartheta^\rho_N \right\|_{\text{Lip}} \, dt \lesssim N^{-1/8} + \left\| \mu_0^N - \vartheta^\rho \right\|_{\text{Lip}}.
\end{equation}

We make the following assumptions on the jump rate function $g : \mathbb{N} \to [0, \infty)$.

(HZRP) The jump rate $g$ satisfies $g(0) = 0, g(n) > 0$ for all $n > 0$, is non-decreasing, uniformly Lipschitz $\sup_{n \geq 0} |g(n+1) - g(n)| < +\infty$, and there are $n_0 > 0$ and $\beta > 0$ such that $g(n') - g(n) \geq \beta$ for any $n' \geq n + n_0$.

The main result on the ZRP is:

\textbf{Theorem 2.1} (Hydrodynamic limit for the ZRP). Consider $\hat{\mathcal{L}}_N$ defined in (2.1) with $g$ satisfying (HZRP). Let $d = 1, f_0 \in C^2(\mathbb{T})$ with $f_0 \geq \delta > 0$, and $\mu_t^N \in P_1(X_N)$ for all $N \geq 1$. Assume $\gamma = 0$, define $\mu_t^N = e^{tN^2\hat{\mathcal{L}}_N}$ and $f_t \in C([0, T], C^3(\mathbb{T}^d))$ solution to (2.4), then the following convergence holds (with quantitative constants)
\begin{equation}
\tag{2.6} \sup_{T \geq 0} \frac{1}{T} \int_0^T \left\| \mu_t^N - \vartheta^\rho_N \right\|_{\text{Lip}} \, dt \lesssim N^{-1/8} + \left\| \mu_0^N - \vartheta^\rho \right\|_{\text{Lip}}.
\end{equation}

\textbf{2.2. The GLK.} In this case, the state space at each site is $X = \mathbb{R}$. Given the choice of a single-site potential $V \in C^2(\mathbb{R})$, the base generator $\hat{\mathcal{L}}_N$ writes
\begin{equation}
\tag{2.7} \hat{\mathcal{L}}_N \Phi(\eta) := \frac{1}{2} \sum_{x \sim y \in \tau_N} \left( \frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right)^2 - \frac{1}{2} \sum_{x \sim y \in \tau_N} \left[ V'(\eta_x) - V'(\eta_y) \right] \left( \frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right)
\end{equation}
where $x \sim y$ neighbours neighbouring sites. The local equilibrium structure is given by
\begin{equation}
\tag{2.8} n_{\lambda}(r) := \frac{e^{\lambda r}}{Z(\lambda)} \quad \text{with} \quad Z(\lambda) := \int_{\mathbb{R}} e^{\lambda r - V(r)} \, dr
\end{equation}
\begin{equation}
\tag{2.9} \sigma \text{ is defined implicitly by } \frac{Z'(\sigma(\rho))}{Z(\sigma(\rho))} \equiv \rho.
\end{equation}
When $\rho \equiv \rho \in \mathbb{R}$ is constant, the local Gibbs measure $\vartheta^\rho_N = \vartheta^\rho_N$ is invariant with average spin $\rho$. The hyperbolic scaling formally leads to zero and the parabolic scaling $\mathcal{L}_N := N^2\hat{\mathcal{L}}_N$ formally leads to
\begin{equation}
\tag{2.10} \partial_t f = 2\Delta [\sigma(f)].
\end{equation}

We assume that the single-site potential satisfies

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The potential $V$ is $C^2$ and decomposes as $V(u) = V_0(u) + V_1(u)$ with $V''_0(u) \geq \kappa$ for all $u \in \mathbb{R}$ for some $\kappa > 0$ and $\| V_1 \|_{W^{1,\infty}(\mathbb{R})} \lesssim 1$.

This assumption is similar with those in [GOVW09, DMOWa, Fat13]. One can take for example a double-well potential, provided it is uniformly convex at infinity.

**Theorem 2.2** (Hydrodynamic limit for the GLK). Consider $\mathcal{L}_N$ defined in (2.6) with $V$ satisfying (HGLK). Let $d = 1$, $f_0 \in C^3(\mathbb{T}^d)$ and $\mu_0^N \in P_1(\mathcal{X}_N)$ for all $N \geq 1$.

Define $\mu^N_t = e^{tN\mathcal{L}_N}$ and $f_t \in C([0, +\infty), C^3(\mathbb{T}^d))$ the global solution to (2.7), then the following convergence holds (with quantitative constants)

$$
\sup_{T \geq 0} \frac{1}{T} \int_0^T \| \mu^N_t - \bar{\vartheta}^N_t \|_{\operatorname{Lip}^*} \, dt \lesssim N^{-1/8} + \| \mu^N_0 - \bar{\vartheta}^N_0 \|_{\operatorname{Lip}^*}.
$$

**3. The abstract strategy**

In this section we sketch the proof of Theorem 1.1. Let $f_t$ be a solution to (1.3).

Given $0 < \ell < N$, we denote by $\eta^\ell$ for the local $\ell$-average $\eta^\ell_N := \sum_{|y-x| \leq \ell} \eta_N$.

Denote by $F^N_t := d\mu_t^N/d\nu_\infty^N$ and $G^N_t := d\bar{\vartheta}^N_t/d\nu_\infty^N$ the densities with respect to $\nu_\infty^N$, and write

$$
\frac{d}{dt} \left( F^N_t - G^N_t \right) = \mathcal{L}_N^* \left( F^N_t - G^N_t \right) + \left( \mathcal{L}_N^* G^N_t - \partial_t G^N_t \right)
$$

so that Duhamel’s formula yields

$$
F^N_t - G^N_t = e^{t \mathcal{L}_N^*} \left( F^N_0 - G^N_0 \right) + \int_0^t e^{(t-s) \mathcal{L}_N^*} \left( \mathcal{L}_N^* G^N_s - \partial_s G^N_s \right) \, ds.
$$

Take $\Phi \in \operatorname{Lip}(\mathcal{X}_N)$ with $\| \Phi \|_{\operatorname{Lip}(\mathcal{X}_N)} \leq 1$ and integrate the above equation to get

$$
\int_{\mathcal{X}_N} \Phi \left( F^N_t - G^N_t \right) \, d\nu_\infty^N
= \int_{\mathcal{X}_N} \left( e^{t \mathcal{L}_N^*} \Phi \right) \left( F^N_0 - G^N_0 \right) \, d\nu_\infty^N + \int_{\mathcal{X}_N} \int_0^t e^{(t-s) \mathcal{L}_N^*} \Phi \left( \mathcal{L}_N^* G^N_s - \partial_s G^N_s \right) \, d\nu_\infty^N \, ds.
$$

(H1) implies $I_1(t) \lesssim \| \mu^N_0 - \bar{\vartheta}^N_0 \|_{\operatorname{Lip}^*}$ and (H3) implies $\frac{1}{T} \int_0^T I_2(t) \, dt \leq c(N) \int_0^T R(s) \, ds$, which implies the conclusion of Theorem 1.1.

**4. Proof for the ZRP**

In this section we sketch the proof of Theorem 2.1) (hydrodynamical limit for the ZRP). Note for this model $\mathcal{L}_N = \mathcal{L}_N^*$ is symmetric with respect to equilibrium measures. Given $f_t \in C^3(\mathbb{T}^d)$ with $f > \delta$, $\delta > 0$, and $\rho := \int_{\mathbb{T}^d} f$, the density of the local Gibbs measure relatively to the invariant measure with mass $\rho$ is

$$
G^N_{\delta}(\eta) := \frac{d\bar{\vartheta}^N(\eta)}{d\bar{\vartheta}^\rho(\eta)} = \prod_{x \in \mathbb{T}^d} \left( \frac{\sigma(f_t(x/N))}{\sigma(\rho)} \right)^{\eta(x)} \left( \frac{Z(\sigma(\rho))}{Z(\sigma(f_t(x/N)))} \right)^{-1}.
$$

where the function $\sigma(r)$ is defined by $\langle n_{\sigma(r)}, \eta(x) \rangle = r$ and the partition function $Z : [0, \lambda^*] \to \mathbb{R}$ is defined in (2.2), with $\lambda^* \in [0, +\infty]$ denoting the radius of convergence of the series.

It is proved in [KL99, Chapter 2, Section 3] that assumption (HZRP) on $g$ implies that $\sigma = R^{-1} : [0, \infty) \to [0, \infty)$ is well-defined and strictly increasing, with

$$
R(\lambda) = \lambda \partial_\lambda \log(Z(\lambda)) = \frac{1}{Z(\lambda)} \sum_{n \geq 0} \frac{n \lambda^n}{\eta(n)!}.
$$
Then the building block $n_\rho$ of the Gibbs measure satisfies $\langle n_{\sigma(\rho)}, g(\eta(x)) \rangle = \sigma(\rho)$. Moreover (HZRP) implies that the function $\sigma$ is $C^\infty$ with uniform bound on all derivatives on $\mathbb{R}_+$, with Lipschitz constant less than $g^*$, see [KL99, Corollary 3.6], and with $\inf_{\lambda > 0} \lambda^{-1} \sigma(\lambda) > 0$ (in particular $\sigma'(0) > 0$). Finally (HZRP) also implies the following comparison principle: if one starts from two ordered configurations $\eta \leq \zeta$ (at all points $x \in \mathbb{T}^d_N$) then the evolution preserves this inequality at later times: $\eta_t \leq \zeta_t$. This implies that if for any $f^N \in C_b(\mathbb{X}_N)$ so that $f^N(\eta) \leq f^N(\zeta)$ for all $\eta \leq \zeta$ one has $\langle \mu_{t-1}^N, f^N \rangle \leq \langle \mu_{0}^N, f^N \rangle$, then at later times $\mu_{t+1}^N < \mu_{t}^N$. It easy to deduce that the the $k$th moments ($k \in \mathbb{N}$)

$$M_k[\mu^N] := \left\langle \mu_t^N, \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta(x)^k \right\rangle$$

are uniformly bounded along time when $\mu_{0}^N < C\theta_{\rho}^N$ for some $C > 0$ and $\rho \in \mathbb{R}_+$.

**4.1. Microscopic Stability – hypothesis (H1).** We use again the “basic coupling” as in [Lig85, Rez91]. We define

$$\tilde{\mathcal{L}}_N \Psi(\eta, \zeta) := \sum_{x,y \in \mathbb{T}^d_N} p(y-x) \left( g(\eta_x) \wedge g(\zeta_x) \right) \left[ \Psi(\eta^{xy}, \zeta^{xy}) - \Psi(\eta, \zeta) \right]$$

for a two-variable test function $\Psi(\eta, \zeta)$. Then $\tilde{\mathcal{L}}_N \Phi(\eta) = \tilde{\mathcal{L}}_N \Phi(\zeta)$ and $\tilde{\mathcal{L}}_N \Phi(x) = \tilde{\mathcal{L}}_N \Phi(\zeta)$, and (H1) follows from the fact that $e^{t\tilde{\mathcal{L}}_N}$ preserves sign and the inequality

$$\tilde{\mathcal{L}}_N \left( \sum_{z \in \mathbb{T}_N^d} |\eta_z - \zeta_z| \right) \leq 0.$$

To prove the latter inequality, we compute

$$\tilde{\mathcal{L}}_N \left( \sum_{z \in \mathbb{T}_N^d} |\eta_z - \zeta_z| \right) = \sum_{x,y \in \mathbb{T}^d_N} p(y-x) \left( g(\eta_x) - g(\eta_x) \wedge g(\zeta_x) \right)$$

$$\times \left[ |\eta^{xy}_x - \zeta_x| + |\eta^{xy}_y - \zeta_y| - |\eta_x - \zeta_x| - |\eta_y - \zeta_y| \right]$$

$$+ \sum_{x,y \in \mathbb{T}^d_N} p(y-x) \left( g(\zeta_x) - g(\eta_x) \wedge g(\zeta_x) \right)$$

$$\times \left[ |\eta_x - \zeta^{xy}_x| + |\eta_y - \zeta^{xy}_y| - |\eta_x - \zeta_x| - |\eta_y - \zeta_y| \right].$$

When $g(\eta_x) - g(\eta_x) \wedge g(\zeta_x) > 0$ necessarily $\eta_x - \zeta_x \geq 1$ and

$$\left[ |\eta^{xy}_x - \zeta_x| + |\eta^{xy}_y - \zeta_y| - |\eta_x - \zeta_x| - |\eta_y - \zeta_y| \right] \leq 0.$$

When $g(\zeta_x) - g(\eta_x) \wedge g(\zeta_x) > 0$ necessarily $\zeta_x - \eta_x \geq 1$ and

$$\left[ |\eta_x - \zeta^{xy}_x| + |\eta_y - \zeta^{xy}_y| - |\eta_x - \zeta_x| - |\eta_y - \zeta_y| \right] \leq 0.$$
4.2. Macroscopic stability – hypothesis (H2). In the parabolic scaling the limit PDE is the nonlinear diffusion equation (2.4). We take $B = C^3$ with its standard infinity Banach norm. The proof that this norm remains uniformly bounded in time is classical in dimension $d = 1$ (using the bounds on $\sigma$), and $f_t \in [\delta, 1 - \delta]$ for all times by maximal principle. Moreover $f_t \to \rho$ exponentially fast as $t \to \infty$ in $B$.

4.3. Consistency estimate – hypothesis (H3). Note that the operator is self-adjoint, $\mathcal{L}_N^* = \mathcal{L}_N$, with respect to the equilibrium measures. We assume $\gamma = 0$.

**Proposition 4.1.** Given $d = 1$ and the solution $f_t \in C^3(\mathbb{T}^d)$ to (2.4) with $f \geq \delta$, $\delta > 0$, and $\rho := \int_{\mathbb{T}^d} f$, and $G_N^s$ defined in (4.1), we have for every $\Phi \in \text{Lip}(X_N)$

$$
\frac{1}{T} \int_0^T I_N^t \text{dt} := \frac{1}{T} \int_0^T \left( e^{(t-s)\mathcal{L}_N} \Phi \right) \left[ \mathcal{L}_N G_N^s - \frac{d}{ds} G_N^s \right] \text{d}v_N^s \text{ds} \text{dt} = O(N^{-1/8})
$$

where the constant depends on the estimates in (H2).

**Proof.** We start by computing

$$
\mathcal{L}_N G_N^s - \frac{d}{ds} G_N^s = \sum_{x \in \mathbb{T}_N^d} A_x^N G_N^s
$$

with (note that $f_t \to \rho$ exponentially fast)

$$
A_x^N := N^2 \sum_{y \in \mathbb{T}_N^d} p(y - x)g(\eta_x) \left( \frac{\sigma(f_t(x/N))}{\sigma(f_t(y/N))} - 1 \right) - \eta_x \frac{\sigma'(f_t(x/N))}{\sigma(f_t(x/N))} \Delta_x \sigma(f_t)(x/N)
$$

$$
= \frac{g(\eta_x)}{\sigma(f_t(x/N))} \Delta_x \sigma(f_t)(x/N) - \eta_x \frac{\sigma'(f_t(x/N))}{\sigma(f_t(x/N))} \Delta_x \sigma(f_t)(x/N) + O(e^{-C_s/N})
$$

for some $C > 0$. Since (conservation of mass)

$$
\int_{X_N} \left( \sum_{x \in \mathbb{T}_N^d} A_x^N G_N^s \right) \text{d}v_N^s = \int_{X_N} \left( \sum_{x \in \mathbb{T}_N^d} A_x^N \right) \text{d}v_N^s = 0,
$$

we can replace $\Phi_{t-s} \mapsto e^{(t-s)\mathcal{L}_N} \Phi$ by

$$
\tilde{\Phi}_{t,s} := e^{(t-s)\mathcal{L}_N} \Phi - E \Phi_{t,s} \left[ e^{(t-s)\mathcal{L}_N} \Phi \right]
$$

and use the Lipschitz bound on $e^{(t-s)\mathcal{L}_N} \Phi$ (microscopic stability) to get

$$
I_N^t = \int_0^t \int_{X_N} \tilde{\Phi}_{t,s} (\eta) \left( \sum_{x \in \mathbb{T}_N^d} A_x^N \right) \text{d}v_N^s \text{dt} + O(1/N)
$$

with $\tilde{A}_x^N$ defined by (note that it has zero average against $\text{d}v_N^s$)

$$
\tilde{A}_x^N := \left\{ g(\eta_x) - \sigma(f_t(x/N)) - \sigma'(f_t(x/N)) \eta_x \right\} \Delta_x \sigma(f_t)(x/N)
$$

We then form sub-sum over non-overlapping cubes of size $\ell \in \{1, \ldots, N\}$ (this intermediate scale factor $\ell$ will be chosen later in terms of $N$). Let $\mathcal{R}_N^d \subset \mathbb{T}_N^d$ be a net of centers of non-overlapping cubes of the form $\mathcal{C}_y := \{ y \in \mathbb{T}_N^d : ||y - x||_\infty \leq \ell \}$. Then

$$
I_N^t = \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{X_N} \tilde{\Phi}_{t,s} (\eta) \left( \sum_{y \in \mathcal{C}_x} \tilde{A}_y^N \right) \text{d}v_N^s \text{dt} + O(1/N)
$$

$$
= (2\ell + 1)^d \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{X_N} \tilde{\Phi}_{t,s} (\eta) \tilde{A}_x^N \text{d}v_N^s \text{dt} + O(1/N)
$$

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with the $\hat{\Lambda}^N_x$ defined by

$$\hat{\Lambda}^N_x := \{\langle g(\eta) \rangle_{C_x} - \sigma(f_t(x/N)) - \sigma'(f_t(x/N)) \langle [\eta]_{C_x} - f_t(x/N) \rangle \} \Delta a[\sigma(f)] (x/N) \}
$$

where $\langle F(\eta) \rangle_{C_x}$, for $F = F(\eta_x)$, denotes taking the average over the cube $C_x$. Note that the average of $\hat{\Lambda}^N_x$ against $d\nu^N_{f_t}$ is $O(e^{-C_N} \ell/N)$. Then

$$\sum_{x \in \mathbb{R}^N_d} \int_0^t \int_{X_N} \Phi_{t,s} \hat{\Lambda}^N_x d\nu^N_{f_t}$$

$$= \sum_{x \in \mathbb{R}^N_d} \int_0^t \int_{X_N} \left( \Phi_{t,s} - \Pi^N_x \Phi_{t,s} \right) \hat{\Lambda}^N_x d\nu^N_{f_t} + \sum_{x \in \mathbb{R}^N_d} \int_0^t \int_{X_N} \Pi^N_x \Phi_{t,s} \hat{\Lambda}^N_x d\nu^N_{f_t}$$

$$= \sum_{x \in \mathbb{R}^N_d} \int_0^t \int_{X_N} \left( \Phi_{t,s} - \Pi^N_x \Phi_{t,s} \right) \hat{\Lambda}^N_x d\nu^N_{f_t}$$

$$+ \sum_{x \in \mathbb{R}^N_d} \int_0^t \int_{X_N} \left( \Pi^N_x \Phi_{t,s} - \mathbb{E}_{\eta_x} \langle \Pi^N_x \Phi_{t,s} \rangle \right) \hat{\Lambda}^N_x d\nu^N_{f_t} =: J^1_t + J^2_t$$

where $\Pi^N_x$ projects on the local equilibrium with same mass in the cube $C_x$ (and does not touch the other site):

$$\Pi^N_x \varphi(\eta) = [\Pi^N_x \varphi](\langle \eta \rangle_{C_x}) = \int_{\Omega_m \setminus \Omega_{\eta_x}} \varphi(\tilde{\eta}) d\nu^{f_t}(\eta_{C_x}) \quad \text{with} \quad \Omega_m := \{ \tilde{\eta} : \langle \tilde{\eta} \rangle_{C_x} = m \}$$

for a function $\varphi$ on $\mathcal{X}^{C_x}$. To estimate the first term $J^1_t$ we first approximate the measure $\nu^N_{f_t}$ on $C_x$ by the equilibrium measure with local mass $f_t(x/N)$, and denote it by $\overline{\nu}_{f_t}$. (note that the approximation is made differently for each cube and depends on $x$, even if it is written explicitly). This produces an error $O(\ell^{d+1}/N)$ (using the Lipschitz regularity of $\Phi_{t,s}$ and the exponential convergence $f_t \rightarrow \rho$ to get uniform in time bounds). We then apply the Poincaré inequality [LSV96, Theorem 1.1] in the cube $C_x$ (whose constant is independent of the number of particles and proportional to the size of the cube) and the law of large number $\| \hat{\Lambda}^N_x \|_{L^2(\overline{\nu}^N_{f_t})} = O(e^{-C_N} \ell^{-d/2})$:

$$J^1_t \leq \sum_{x \in \mathbb{R}^N_d} \int_0^t \| \Phi_{t,s} - \Pi^N_x \Phi_{t,s} \|_{L^2(\overline{\nu}^N_{f_t})} \| \hat{\Lambda}^N_x \|_{L^2(\overline{\nu}^N_{f_t})} ds + O(\ell^{d+1}/N)$$

$$\lesssim \ell^{1-d/2} \sum_{x \in \mathbb{R}^N_d} \int_0^t \sqrt{D^\ell_x(\Phi_{t,s})} e^{-C_x} ds + O(\ell^{d+1}/N)$$

$$\lesssim \ell^{1-d/2} N^{d/2} \int_0^t \left( \sum_{x \in \mathbb{R}^N_d} D^\ell_x(\Phi_{t,s}) \right)^{1/2} e^{-C_x} ds + O(\ell^{d+1}/N)$$

where $D^\ell_x(\Phi)$ is the Dirichlet form on the cube $C_x$ with respect to the measure $\overline{\nu}_{f_t}$:

$$D^\ell_x(\Phi) := \sum_{y, z \in \mathbb{Z}^{C_x}} \int_{X_N} p(z - y) g(\eta_y) \left[ \Phi(\eta^{yz}) - \Phi(\eta)^2 \right] d\overline{\nu}^N_{f_t}.$$
where the last error accounts for the jumps at the border between two cubes. We deduce that
\[
\int_0^T J_1^t \, dt \lesssim T^{1/2} (\ell/N)^{1-d/2} + O (T \ell^{d+1}/N).
\]

To control the second term \(J_2^t\), we first use the equivalence of ensemble in [KL99, Appendix II, Corollary 1.7] on the measure \(\mathcal{P}_{f_s}^N\) (together with the exponential tail estimates on the local Gibbs measure) to get
\[
\langle g(\eta) \rangle_{\mathcal{C}_z} = \sigma (\langle \eta \rangle_{\mathcal{C}_z}) + O (1/\ell^d).
\]

Second we remark that the Lipschitz regularity of \(\Phi_{t-s}\) implies that
\[
\Pi_x^N \Phi_{t-s} - E_{\theta_{f_s}^N} [\Pi_x^N \Phi_{t-s}] = O (\ell^d N^{-d}),
\]
and since the average of \(\mathcal{A}_x^N\) with respect to \(\theta_{f_s}^N\) is \(O (\ell/N)\), we can write
\[
J_2^t = \sum_{x \in \mathbb{R}_N^d} \int_0^t \int_{X_N} (\Pi_x^N \Phi_{t-s} (\eta))_{\mathcal{C}_z} - \Pi_x^N \Phi_{t-s} (f_s (x/N)) \right) \mathcal{A}_x^N \, d\theta_{f_s}^N + O (\ell/N).
\]

Third, we remark that the Lipschitz regularity of \(\Phi_{t-s}\) (with constant \(N^{-d}\)) implies a Lipschitz regularity of its averaged projection \(\Pi_x^N \Phi_{t-s}\) with constant \(\ell^d N^{-d}\), with respect to the local mass. Indeed, given \(0 = m \leq m' < +\infty\), pick any pair of configuration \((\eta_0, \zeta_0)\) with \(\langle \eta_0 \rangle_{\mathcal{C}_z} = m, \langle \zeta_0 \rangle_{\mathcal{C}_z} = m'\) and \(\eta_0 \leq \zeta_0\) (such configuration trivially exists since \(m \leq m'\)). Then define the initial trivial coupling \(\delta_{(\eta_0, \zeta_0)}\) on \(\Omega_m \times \Omega_{m'}\), and \(\Pi := \lim_{t \to \infty} e^{t \mathcal{L}_N} \delta_{(\eta_0, \zeta_0)}\). The measure \(\Pi\) has marginals \(\nu^{\ell,m}\) and \(\nu^{\ell,m'}\) (convergence to equilibrium), and since the initial coupling only charges configurations \((\eta, \zeta)\) with \(\eta \leq \zeta\), the same is true of the limiting coupling (note that our coupling preserves the ordering), and
\[
\Pi_x^N \Phi_{t-s} (m') - \Pi_x^N \Phi_{t-s} (m) = \int_{\Omega_{m'}} \Phi_{t-s} (\eta') \, d\nu^{\ell,m'} (\eta') - \int_{\Omega_m} \Phi_{t-s} (\eta) \, d\nu^{\ell,m} (\eta)
\]
\[
= \int_{\Omega_{m'} \times \Omega_m} [\Phi_{t-s} (\eta') - \Phi (\eta)] \, d\Pi (\eta, \eta')
\]
and since \(\eta \leq \eta'\) on the support of \(\Pi\), \(\|\eta' - \eta\|_{\ell; (\mathcal{C}_z)} \leq (m' - m) \ell^d N^{-d}\) and
\[
\|\Pi_x^N \Phi_{t-s} (m') - \Pi_x^N \Phi_{t-s} (m)\| \leq \frac{\ell^d}{N^d} |m' - m|.
\]

We deduce (using (5.3))
\[
J_2^t \lesssim \frac{\ell^d}{N^d} \sum_{x \in \mathbb{R}_N^d} \int_0^t \int_{X_N} |\langle \eta \rangle_{\mathcal{C}_z} - f_s (x/N)| \times
|\sigma (\langle \eta \rangle_{\mathcal{C}_z}) - \sigma (f_s (x/N))| \times |\langle \eta \rangle_{\mathcal{C}_z} - f_s (x/N)| \, d\theta_{f_s}^N e^{-Cs} \, ds
+ \frac{1}{N^d} \sum_{x \in \mathbb{R}_N^d} \int_0^t \int_{X_N} |\langle \eta \rangle_{\mathcal{C}_z} - f_s (x/N)| \, d\theta_{f_s}^N e^{-Cs} \, ds + O (\ell/N)
\]
which yields by Taylor formula, the approximation of \( \vartheta^N_t \) by \( \vartheta^N_{f_s} \), and the law of large numbers

\[
J^2_t \lesssim \frac{e^d}{N^d} \sum_{x \in \mathbb{R}^d_N} \int_0^t \int_{X^N} |\langle n \rangle c_x - f_s(x/N)|^3 \, d\vartheta^N_t e^{-Cs} \, ds + \frac{1}{N^d} \sum_{x \in \mathbb{R}^d_N} \int_0^t \int_{X^N} |\langle n \rangle c_x - f_s(x/N)| \, d\vartheta^N_t e^{-Cs} \, ds + O(\ell/N) e^{-Cs} \, ds
\]

\[
+ \frac{1}{N^d} \sum_{x \in \mathbb{R}^d_N} \int_0^t \int_{X^N} |\langle n \rangle c_x - f_s(x/N)| \, d\vartheta^N_{f_s} e^{-Cs} \, ds + O(\ell/N)
\]

\[
\lesssim O\left(\ell^{-3d/2}\right) + O(\ell/N).
\]

Combining all estimates we get (optimizing \( \ell := N^{1/4} \))

\[
\frac{1}{T} \int_0^T I^N_t \, dt \lesssim \left( \frac{1}{N} + \frac{\ell^{1+d/2}}{N^{1-d/2}} + \frac{\ell^{1+2d}}{N} + \frac{1}{N} + \frac{\ell}{N} \right) \lesssim \frac{1}{N^{1/8}}.
\]

5. **Proof for the GLK**

In this section we prove Theorem 2.2. Note again that for this model \( \mathcal{L}_N = \mathcal{L}^*_N \) is symmetric with respect to equilibrium measures. Given \( f_t \in C^3(\mathcal{T}^d) \) and \( \rho := f_{T_n} \in \mathbb{R} \), the density of the local Gibbs measure relatively to the invariant measure with mass \( \rho \) is:

\[
G^N_t(\eta) := \frac{d\vartheta^N_t(\eta)}{d\varrho^N_t(\eta)} = \prod_{x \in \mathbb{T}^N_d} e^{[\sigma(\eta(x/N)) - \sigma(\rho)]\eta_x} \frac{Z(\sigma(\rho))}{Z(\sigma(f_t(x/N)))}.
\]

where the function \( \sigma(r) \) is defined by \( \langle n_{\sigma(r)}, \eta \rangle = r \) and the partition function \( Z(\lambda) = \int_{\mathbb{R}} e^{\lambda r - V(r)} \, dr \) is defined on \( \mathbb{R} \). The uniform convexity of \( V \) at infinity easily implies bounds on all moments of the invariant measure

\[
\int_{X^N} \sum_{x \in \mathbb{T}^N_d} \eta(x)^k d\varrho^N_t(\eta) = C_k < \infty.
\]

and it follows from classical estimates with the Legendre transform that there exists \( C > 0 \) so that \( 0 < \frac{1}{2} \leq \sigma' \leq C < \infty \) (see [GOVW09, Lemma 41] and [DMOWa, Lemma 5.1]).

5.1. **Microscopic stability — hypothesis (H1)**. We consider a coupling of two Ginzburg-Landau processes with generator \( \tilde{L}_N : C_b(\mathbb{X}^3_N) \to C_b(\mathbb{X}^3_N) \) given by

\[
\tilde{L}_N \Psi(\eta, \zeta) := \sum_{x \sim y} \left( \left[ \left( \frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right)^* \left( \frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right) \right] \otimes 1 \right) \Psi(\eta, \zeta)
\]

\[
+ \left[ 1 \otimes \left( \frac{\partial}{\partial \zeta_x} - \frac{\partial}{\partial \zeta_y} \right)^* \left( \frac{\partial}{\partial \zeta_x} - \frac{\partial}{\partial \zeta_y} \right) \right] \Psi(\eta, \zeta)
\]

\[
+ (2 + K) \left( \frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right) \otimes \left( \frac{\partial}{\partial \zeta_x} - \frac{\partial}{\partial \zeta_y} \right) \Psi(\eta, \zeta)
\]

\[
(5.2)
\]
where $K$ is a constant to be chosen later and the adjoint is taken in $L^2(d\nu^N_{\rho})$ so
\[
\hat{L}^N = \sum_{x \sim y} \left( \frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right)^2 - (V'(\eta_x) - V'(\eta_y)) \left( \frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right).
\]

Then for any $p \in (1, 2]$ there is $K > 0$ (depending on $p$) so that
\[
\tilde{L}^N \left( \sum_x |\eta_x - \zeta_x|^p \right) \leq 0.
\]

This implies the weak contraction of the evolution in $W^p_N$ ($p$-Wasserstein distance) for any $p \in (1, 2]$, and thus by limit in $W^1_N$. By duality this implies that the evolution is weakly contractive for the dual Lipschitz norm.

5.2. **Macroscopic stability - hypothesis (H2).** The limit equation is a one-dimensional nonlinear diffusion equation with uniform ellipticity bounds, and standard elliptic theory shows that the solution exists globally and converges exponentially fast to a constant in $C^3(\mathbb{T}^d)$.

5.3. **Consistency estimate - hypothesis (H3).**

**Proposition 5.1.** Given $d = 1$ and the solution $f_t \in C^3(\mathbb{T}^d)$ to (2.4), and $\rho := \int_{\mathbb{T}^d} f$, and $G^N_t$ defined in (5.1), we have for every $\Phi \in \text{Lip}(X_N)$
\[
\frac{1}{T} \int_0^T I^N_T dt := \frac{1}{T} \int_0^T \left( e^{(t-s)\hat{L}^N} \Phi \right) \left( \mathcal{L}^N G^N_s - \frac{d}{ds} G^N_s \right) \, d\nu^N_s \, dt = O \left( N^{-1/8} \right)
\]

where the constant depends on the estimates in (H2).

**Proof.** We start by computing
\[
\mathcal{L}^N G^N_s - \frac{d}{ds} G^N_s = \sum_{x \in \mathbb{T}^d_N} A^N_x G^N_s
\]

with (note that $f_t \to \rho$ exponentially fast)
\[
A^N_x := \frac{N^2}{2} \sum_{y \sim x} \left[ (\sigma(f_s(x/N)) - \sigma(f_s(y/N)))^2 - (V'(\eta_x) - V'(\eta_y)) (\sigma(f_s(x/N)) - \sigma(f_s(y/N))) \right]
\]
\[
- \sum_x (\eta_x - f_s(x/N)) \sigma'(f_s(x/N)) \Delta[\sigma(f)](x/N)
\]
\[
\frac{N^2}{2} \sum_{y \neq x} \left[ 2\sigma(f_s(x/N)) (\sigma(f_s(x/N)) - \sigma(f_s(y/N))) 
- 2V'(\eta_x)\sigma(f_s(x/N)) - \sigma(f_s(y/N)) \right] 
- \sum_x (\eta_x - f_s(x/N)) \sigma'(f_s(x/N)) \Delta[\sigma(f)](x/N) 
= \Delta[\sigma(f)](x/N) \left[ V'(\eta_x) - \sigma(f_s(x/N)) \right] 
- \sigma'(f_s(x/N)) (\eta_x - f_s(x/N)) + O(1/N)
\]
for some \(C > 0\). Since (conservation of mass)
\[
\int_{\mathcal{X}_N} \sum_{x \in \mathcal{Y}_d^N} A_x^N G_s^N \, d\nu_N = \int_{\mathcal{X}_N} \sum_{x \in \mathcal{Y}_d^N} A_x^N \, d\theta_{f_s}^N = 0,
\]
we can replace \(\Phi_{t-s} := e^{(t-s)\mathcal{L}_N} \Phi\) by
\[
\tilde{\Phi}_{t,s} := e^{(t-s)\mathcal{L}_N} \Phi - E_{f_s} \left[ e^{(t-s)\mathcal{L}_N} \Phi \right]
\]
and use the Lipschitz bound on \(e^{(t-s)\mathcal{L}_N} \Phi\) (microscopic stability) to get
\[
I_t^N = \int_0^t \int_{\mathcal{X}_N} \tilde{\Phi}_{t,s}(\eta) \left( \sum_{x \in \mathcal{Y}_d^N} \tilde{A}_x^N \right) \, d\theta_{f_s}^N + O(1/N)
\]
with \(\tilde{A}_x^N\) defined by (note that it has zero average against \(d\theta_{f_s}^N\))
\[
\tilde{A}_x^N := \Delta[\sigma(f)](x/N) \left[ V'(\eta_x) - \sigma(f_s(x/N)) - \sigma'(f_s(x/N)) (\eta_x - f(x/N)) \right].
\]
We then form sub-sum over non-overlapping cubes of size \(\ell \in \{1, \ldots, N\}\) (this intermediate scale factor \(\ell\) will be chosen later in terms of \(N\)). Let \(\mathcal{R}_N^d \subset \mathcal{T}_N^d\) be a net of centers of non-overlapping cubes of the form \(\mathcal{C}_x := \{y \in \mathcal{T}_N^d : \|x - y\|_{\infty} \leq \ell\}\). Then
\[
I_t^N = \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathcal{X}_N} \tilde{\Phi}_{t,s}(\eta) \left( \sum_{y \in \mathcal{C}_x} \tilde{A}_y^N \right) \, d\theta_{f_s}^N + O(1/N)
\]
\[
= (2\ell + 1)^d \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathcal{X}_N} \tilde{\Phi}_{t,s}(\eta) \tilde{A}_x^N \, d\theta_{f_s}^N + O(1/N)
\]
with the \(\tilde{A}_x^N\) defined by
\[
\tilde{A}_x^N := \Delta[\sigma(f)](x/N) \left[ (V'(\eta))_{\mathcal{C}_x} - \sigma(f(x/N)) - \sigma'(f_s(x/N)) (\langle \eta \rangle_{\mathcal{C}_x} - f(x/N)) \right]
\]
where \( \langle F(\eta) \rangle_{C_x} \), for \( F = F(\eta_x) \), denotes taking the average over the cube \( C_x \). Note that the average of \( \tilde{A}_{\epsilon}^N \) against \( d\tilde{\vartheta}_{f_s}^N \) is \( O(e^{-C_s \ell/N}) \). Then

\[
\sum_{x \in \mathbb{R}_N^d} \int_0^t \int_{X_N} \tilde{\Phi}_{t,s} \tilde{A}_{\epsilon}^N d\tilde{\vartheta}_{f_s}^N \\
= \sum_{x \in \mathbb{R}_N^d} \int_0^t \int_{X_N} (\Phi_{t,s} - \Pi_x^N \tilde{\Phi}_{t,s}) \tilde{A}_{\epsilon}^N d\tilde{\vartheta}_{f_s}^N + \sum_{x \in \mathbb{R}_N^d} \int_0^t \int_{X_N} \Pi_x^N \tilde{\Phi}_{t,s} \tilde{A}_{\epsilon}^N d\tilde{\vartheta}_{f_s}^N \\
= \sum_{x \in \mathbb{R}_N^d} \int_0^t \int_{X_N} (\Phi_{t,s} - \Pi_x^N \Pi_x^{-1}) \tilde{A}_{\epsilon}^N d\tilde{\vartheta}_{f_s}^N \\
+ \sum_{x \in \mathbb{R}_N^d} \int_0^t \int_{X_N} \left( \Pi_x^N \Phi_{t,s} - E_{\partial_x (\Pi_x^{-1} \Pi_x^N) f_s} \right) \tilde{A}_{\epsilon}^N d\tilde{\vartheta}_{f_s}^N =: J_1^1 + J_1^2
\]

where \( \Pi_x^N \) projects on the local equilibrium with same spin in the cube \( C_x \) (and does not touch the other site):

\[
\Pi_x^N \varphi(\eta) = [\Pi_x^N \varphi](\eta)_{C_x} = \int_{\Omega(\eta)_{C_x}} \varphi(\eta) d\vartheta^{f_s(\eta)C_x}(\eta) \quad \text{with} \quad \Omega_m := \{ \eta : \langle \eta \rangle_{C_x} = m \}
\]

for a function \( \varphi \) on \( X^C \). To estimate the first term \( J_1^1 \) we first approximate the measure \( \tilde{\vartheta}_{f_s}^N \) on \( C_x \) by the equilibrium measure with local mass \( f_s(x/N) \), and denote it by \( \bar{\vartheta}_{f_s} \) (note that the approximation is made differently for each cube and depends on \( x \), even if it is written explicitly). This produces an error \( O(\ell^d/2N) \) (using the Lipschitz regularity of \( \Phi_{t,s} \) and the exponential convergence \( f_s \to \rho \) to get uniform in time bounds). We then apply the Poincaré inequality [LY93, Theorem 2] in the cube \( C_x \) (whose constant is independent of the number of particles and proportional to the size of the cube) and the law of large number \( \| \tilde{A}_{\epsilon}^N \|_{L^2(\bar{\vartheta}_{f_s}^N)} = O(e^{-C_s \ell/d}) \):

\[
J_1^1 \leq \sum_{x \in \mathbb{R}_N^d} \int_0^t \| \Phi_{t,s} - \Pi_x^N \Pi_x^{-1} \|_{L^2(\bar{\vartheta}_{f_s}^N)} \| \tilde{A}_{\epsilon}^N \|_{L^2(\bar{\vartheta}_{f_s}^N)} ds + O(\ell^d/2N)
\]

\[
\lesssim \ell^1/d^2 \sum_{x \in \mathbb{R}_N^d} \int_0^t \sqrt{D_x^\ell(\Phi_{t,s})} e^{-Cs} ds + O(\ell^d/2N)
\]

\[
\lesssim \ell^{1-d/2} N^{d/2} \int_0^t \left( \sum_{x \in \mathbb{R}_N^d} D_x^\ell(\Phi_{t,s}) \right)^{1/2} e^{-Cs} ds + O(\ell^d/2N)
\]

where \( D_x^\ell(\Phi) \) is the Dirichlet form on the cube \( C_x \) with respect to the measure \( \bar{\vartheta}_{f_s}^N \):

\[
D_x^\ell(\Phi) := \sum_{y \sim x \in \mathbb{C}_x} \int_{X_N} \left[ \partial_{\theta_x} \Phi(y) - \partial_{\theta_y} \Phi(y) \right]^2 d\bar{\vartheta}_{f_s}^N.
\]

Then we compute

\[
\frac{1}{2N^2} \frac{d}{dt} \int_{X_N} \Phi_{t,s}(\eta)^2 d\bar{\vartheta}_{f_s}^N \leq - \sum_{x \in \mathbb{R}_N^d} D_x^\ell(\Phi_{t,s}) + O(1/N^2)
\]

where the last error accounts for the jumps at the border between two cubes. We deduce that

\[
\int_0^T J_1^1 dt \lesssim T^{1/2} \ell^{1-d/2} + O(T^d/2N)
\]
To control the second term $J_t^2$, we first use the equivalence of ensemble in [LPY02, Corollary 5.3] on the measure $\tilde{\nu}_f^N$ (together with the exponential tail estimates on the local Gibbs measure) to get

\begin{equation}
    \langle V'(\eta) \rangle_{\mathcal{C}_x} = \sigma (\langle \eta \rangle_{\mathcal{C}_x}) + O \left( \frac{1}{\ell^d} \right).
\end{equation}

Second, we remark that the Lipschitz regularity of $\Phi_{t-s}$ implies that

$$\Pi_x^N \Phi_{t-s} - \mathbb{E}_{\nu_x^N} [\Pi_x^N \Phi_{t-s}] = O(\ell^d N^{-d}),$$

and since the average of $\hat{A}_x^N$ with respect to $\nu_x^N$ is $O(\ell/N)$, we can write

$$J_t^2 = \sum_{x \in \mathbb{R}_N^d} \int_0^t \int_{\mathbb{R}_N} \left( \Pi_x^N \Phi_{t-s} [\langle \eta \rangle_{\mathcal{C}_x}] - \Pi_x^N \Phi_{t-s} [f_s (x/N)] \right) \mathcal{A}_x \, d\nu_x^N + O \left( \frac{\ell}{N} \right).$$

Third, we remark that the Lipschitz regularity of $\Phi_{t-s}$ (with constant $N^{-d}$) implies a Lipschitz regularity of its averaged projection $\Pi_x^N \Phi_{t-s}$ with constant $\ell^d N^{-d}$, with respect to the local mass. Indeed, given $0 = m < m' < +\infty$, pick any pair of configurations $(\eta_0, \zeta_0)$ with $\langle \eta_0 \rangle_{\mathcal{C}_x} = m$, $\langle \zeta_0 \rangle_{\mathcal{C}_x} = m'$ and $\eta_0 < \zeta_0$ (such configuration trivially exists since $m < m'$). Then consider an initial coupling on $\Omega_m \times \Omega_{m'}$ as a product of localised smooth distribution around $\delta_\eta$ and $\delta_\zeta$, so the support contains only strictly ordered $\eta < \zeta$. Then define $\Pi := \lim_{t \to -\infty} e^{t \mathcal{L}_N} \delta_{(\eta_0, \zeta_0)}$. The measure $\Pi$ has marginals $\nu^{\ell,m}$ and $\nu^{\ell,m'}$ (convergence to equilibrium), and since our coupling satisfies $e^{t \mathcal{L}_N} (\sum_x (\eta_x - \zeta_x)_+^2) \leq 0$, we have that the ordering $\Psi(\eta, \zeta) \leq C \sum_x (\eta_x - \zeta_x)_+^2$ is propagated in time for any $C > 0$ ($\Psi$ here is the evolving coupling), so the limit coupling still has support included in $\eta \leq \zeta$, and

$$\Pi_x^N \Phi_{t-s} (m') - \Pi_x^N \Phi_{t-s} (m) = \int_{\Omega_{m'}} \Phi_{t-s} (\eta') \, d\nu^{\ell,m'} (\eta') - \int_{\Omega_m} \Phi_{t-s} (\eta) \, d\nu^{\ell,m} (\eta)$$

$$= \int_{\Omega_m \times \Omega_{m'}} [\Phi_{t-s} (\eta') - \Phi (\eta)] \, d\Pi (\eta, \eta')$$

and since $\eta \leq \eta'$ on the support of $\Pi$, $\| \eta' - \eta \|_{\ell^1(\mathcal{C}_x)} \leq (m' - m) \ell^d N^{-d}$ and

$$| \Pi_x^N \Phi_{t-s} (m') - \Pi_x^N \Phi_{t-s} (m) | \leq \frac{\ell^d}{N^d} |m' - m|.$$

We deduce (using (5.3))

$$J_t^2 \lesssim \frac{\ell^d}{N^d} \sum_{x \in \mathbb{R}_N^d} \int_0^t \int_{\mathbb{R}_N} |\langle \eta \rangle_{\mathcal{C}_x} - f_s (x/N)| \times$$

$$| \sigma (\langle \eta \rangle_{\mathcal{C}_x}) - \sigma (f_s (x/N)) - \sigma' (f_s (x/N)) [\langle \eta \rangle_{\mathcal{C}_x} - f_s (x/N)] | \, d\nu_x^N e^{-Cs} \, ds$$

$$+ \frac{1}{N^d} \sum_{x \in \mathbb{R}_N^d} \int_0^t \int_{\mathbb{R}_N} |\langle \eta \rangle_{\mathcal{C}_x} - f_s (x/N)| \, d\nu_x^N e^{-Cs} \, ds + O \left( \frac{\ell}{N} \right)$$
which yields by Taylor formula, the approximation of $\vartheta_N f_s$ by $\tilde{\vartheta}_N f_s$, and the law of large numbers

$$J_t^2 \overset{\ell d}{\sim} \frac{\ell d}{N^d} \sum_{x \in \mathbb{R}^d_N} \int_0^t \int_{X_N} |\eta| c_x - f_s(x/N)|^3 \, d\tilde{\vartheta}_N f_s \, e^{-Cs} \, ds$$

$$+ \frac{1}{N^d} \sum_{x \in \mathbb{R}^d_N} \int_0^t \int_{X_N} |\eta| c_x - f_s(x/N)| \, d\tilde{\vartheta}_N f_s \, e^{-Cs} \, ds + O(\ell/N) e^{-Cs} \, ds$$

$$\overset{\ell d}{\sim} \frac{\ell d}{N^d} \sum_{x \in \mathbb{R}^d_N} \int_0^t \int_{X_N} |\eta| c_x - f_s(x/N)|^3 \, d\tilde{\vartheta}_N f_s \, e^{-Cs} \, ds$$

$$+ \frac{1}{N^d} \sum_{x \in \mathbb{R}^d_N} \int_0^t \int_{X_N} |\eta| c_x - f_s(x/N)| \, d\tilde{\vartheta}_N f_s \, e^{-Cs} \, ds + O(\ell/N)$$

$$\overset{\ell d}{\sim} O\left(\ell^{-3d/2}\right) + O(\ell/N).$$

Combining all estimates we get (optimizing $\ell := N^{1/4}$)

$$\frac{1}{T} \int_0^T I_t^N \, dt \lesssim \left(\frac{1}{N} + \frac{\ell^{1+d/2}}{N^{1-d/2}} + \frac{\ell^{1+2d}}{N} + \frac{1}{\ell^{d/2}} + \frac{\ell}{N}\right) \lesssim \frac{1}{N^{1/8}}.$$

**REFERENCES**


