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Averaging and mixing for random perturbations of elliptic equilibria

Sergei Kuksin*

1 Introduction

Consider a small perturbation of an elliptic equilibrium:

$$\begin{aligned} \dot{x}_j - \lambda_j y_j &= \varepsilon \dots, \quad j = 1, \dots, n, \\ \dot{y}_j + \lambda_j x_j &= \varepsilon \dots, \quad j = 1, \dots, n, \end{aligned} \tag{1.1}$$

where $\{\lambda_j\}$ are non-zero real numbers. The problem to study the behaviour of solutions for various perturbations $\varepsilon \dots$, is a classical question of Dynamical Systems. For analytical Hamiltonian perturbations which are $O|(x,y)|^3$ and when all λ_j's are of the same sign the stability of the actions $I_j = (x_j^2 + y_j^2)/2$ during exponentially long (in terms of ε) time is a result of L. Niederman [5], while for non-Hamiltonian perturbations the behaviour of solutions for (1.1) is prescribed by the Krylov–Bogolyubov theory, but only for time-intervals of order ε^{-1}, see [2, 1]. The behaviour of solutions for non-Hamiltonian equations (1.1) during time-intervals significantly longer that ε^{-1} is a hard problem with only a few isolated results available. The goal of my lecture is to discuss the long-time behaviour of solutions for eq. (1.1) when $\varepsilon \dots$ stands for a stochastic perturbation, following the recent review-paper [3]. I will explain that:
– the theory of stochastic equations (1.1) is "more final" then its deterministic counterpart, with easier and more general proofs;
– the property of mixing in stochastic equations (1.1) (which allows a convenient sufficient condition) helps to explain and understand the nature of long-time behaviour of its solutions;
– an analogous theory is available for stochastic PDEs, with similar proofs.

Let us start with an example:

Example 1.1. Consider system (1.1) with $n = 1$, when the perturbation is a Hamiltonian term plus a noise:

$$\begin{aligned} \dot{x} - \lambda y &= -\varepsilon h_y(x,y) + \sqrt{\varepsilon}\,(d/dt)\beta_1(t), \\ \dot{y} + \lambda x &= \varepsilon h_x(x,y) + \sqrt{\varepsilon}\,(d/dt)\beta_2(t), \end{aligned} \tag{1.2}$$

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where β_1, β_2 are standard independent Wiener processes and h is a C^2-function, whose C^2-norm is bounded by 1. Denote $(x, y) = z$. The non-stochastic part of the equation above is Hamiltonian with the Hamiltonian $H = -|z|^2/2 + \varepsilon h(z)$. Applying the Ito formula to $-H(z(t))$ and taking the expectation we find that for small ε, $(d/dt)\mathbb{E}(-H(z(t))) \geq \varepsilon/4$. So

$$\mathbb{E}|z(t)|^2 \geq \mathbb{E}(-H(z(t))) - \varepsilon \geq \tfrac{1}{4}\varepsilon t - \text{Const}.$$

We conclude that:

1) using perturbative methods we can study solutions of general stochastic equations (1.1) only for $t \lesssim \varepsilon^{-1}$;

2) in order to be able to examine perturbativly such equations for $t \gg \varepsilon^{-1}$ we must add friction to the system.

But the added friction "stabilises everything" and thus simplifies the problem of studying stochastic equations (1.1) for long time.

2 Averaging

Denote $x_j + iy_j = v_j$. Then $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ and stochastic eq. (1.1) with (the easiest possible) additive noise reeds

$$\frac{d}{dt}v_k + i\lambda_k v_k = \varepsilon P_k(v) + \sqrt{\varepsilon}\,\frac{d}{dt}b_k(\beta_k^1 + i\beta_k^2)(t), \quad k = 1, \dots, n,$$

where b_k are some real numbers and β_k^1, β_k^2 are standard independent Wiener processes. What I will say below stays true for equations with general additive noises on the price of heavier formulas; and – with minimal changes - for equations with non-additive noises, depending on v; see [3, Section 8]. The processes $\beta_k^c := \beta_k^1 + i\beta_k^2$ are called *standard complex Wiener processes*. Passing to the slow time

$$\tau = \varepsilon t$$

we re-write the system as

$$\dot v_k(\tau) + i\varepsilon^{-1}\lambda_k v_k = P_k(v) + b_k\dot\beta_k^c(\tau), \quad k = 1, \dots, n, \quad v(0) = v_0. \tag{2.1}$$

Here and below the upper dot stands for $d/d\tau$, and we denoted by the same notation β_k^c another set of standard independent complex Wiener processes, obtained by a proper scaling of the original processes.

Below we assume that eq. (2.1) satisfies the following assumption, where $T > 0$ is fixed:

(A1) 1) the mapping $P : \mathbb{C}^n \to \mathbb{C}^n$ is locally Lipschitz, and

$$\begin{aligned} |P(v)| &\leq C_P(1 + |v|)^{m_0}, \\ \text{Lip}(P|_{\{|v|\leq R\}}) &\leq C_P(1 + R)^{m_0} \ \ \forall R > 0, \end{aligned} \tag{2.2}$$

for some $m_0 \in \mathbb{N}$ and $C_P > 0$.

2) Eq. (2.1) has a unique solution $v^\varepsilon(\tau; v_0)$, $0 \le \tau \le T$, and

$$\mathbb{E} \sup_{\tau\in[t,(t+1)\wedge T]} |v^\varepsilon(\tau; v_0)|^{2m'} \le C_{m'}(|v_0|) \qquad \forall\, 0 \le t \le T-1, \tag{2.3}$$

for some $m' > m_0$.

2.1 Interaction representation

Denote $\Lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, and for $t \in \mathbb{R}$ consider the linear operators

$$e^{it\Lambda} = \operatorname{diag}\{e^{it\lambda_j}, 1 \le j \le n\} : \mathbb{C}^n \to \mathbb{C}^n.$$

They form a group of linear unitary transformations of $\mathbb{C}^n$. Substituting in (2.1)

$$v(\tau) = e^{-i\tau\varepsilon^{-1}\Lambda} a(\tau)$$

we get for $a(\tau)$ the equation

$$\dot a(\tau) = e^{i\tau\varepsilon^{-1}\Lambda} P(e^{-i\tau\varepsilon^{-1}\Lambda} a(\tau)) + \operatorname{diag}\{b_k\}\dot\beta^c(\tau), \quad a(0) = v_0, \tag{2.4}$$

where $\beta^c = (\beta_1, \dots, \beta_n)$ is yet another set of standard independent complex Wiener processes. Obviously $|a_j(\tau)| \equiv |v_j(\tau)|$ for each j, but the angles of $a_j(\tau)$ and $v_j(\tau)$ differ by $\tau\varepsilon^{-1}\lambda_j$.

We are concerned with the distributions of solutions $a^\varepsilon(\tau)$ for equations (2.4) and of (2.1), i.e. with measures $\mathcal{D}(a^\varepsilon(\tau)) \in \mathcal{P}(\mathbb{C}^n)$, $0 \le \tau \le T$, and $\mathcal{D}(a^\varepsilon(\cdot)) \in \mathcal{P}(C(0, T; \mathbb{C}^n))$, and with similar objects for solutions $v^\varepsilon(\tau)$ of eq. (2.1). Here and below $\mathcal{D}(\cdot)$ signifies the distribution of a random variable, and $\mathcal{P}(M)$ – a set of probability Borel measures on a metric space M.

Since due to the form of eq. (2.1) derivatives in τ of solutions a^ε are of order one uniformly in ε, then evoking the Prokhorov theorem we easily get

Lemma 2.1. *The set of measures* $\{\mathcal{D}(a^\varepsilon(\cdot)), 0 < \varepsilon \le 1\}$ *is precompact in* $\mathcal{P}(C(0, T; \mathbb{C}^n))$ *with respect to the weak topology.*

So every sequence $\varepsilon'_j \to 0$ contains a subsequence $\varepsilon_j \to 0$ such that

$$\mathcal{D}(a^{\varepsilon_j}) \rightharpoonup Q^0 \quad \text{as} \quad \varepsilon_j \to 0, \tag{2.5}$$

for some measure $Q^0 \in \mathcal{P}(C(0, T; \mathbb{C}^n))$ (here $\rightharpoonup$ signifies the weak convergence of measures). Our first goal is to show that the measure Q^0 does not depend on the sequence $\varepsilon_j \to 0$ and explain how to find it.

2.2 Resonant averaging

Motivated by the form of the non-autonomous vector field in (2.4), for any $a \in \mathbb{C}^n$ let us consider the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T e^{it\Lambda} P(e^{-it\Lambda} a) = \langle\!\langle P \rangle\!\rangle(a). \tag{2.6}$$

Lemma 2.2. *The limit* $\langle\!\langle P\rangle\!\rangle(a)$ *exists. This is a locally Lipschitz vector field on* $\mathbb{C}^n$*, satisfying* (2.2) *with the same* m_0 *and* C_P *as* P.

The operator $P \mapsto \langle\!\langle P\rangle\!\rangle$ possesses many nice and natural properties; see [3, Section 3].

Now we define an *effective equation* for eq. (2.4) as the stochastic differential equation

$$\dot{a}(\tau) = \langle\!\langle P\rangle\!\rangle(a) + \operatorname{diag}\{b_k\}\dot{\beta}^c(\tau), \quad a(0) = v_0. \tag{2.7}$$

(Due to the simplicity of the noice term in eq. (2.4) it stays the same in the effective equation. More complicated noises are transformed under transition to an effective equation by an averaging, similar to (2.6), see in [3, Section 8].)

Due to the lemma above a solution of (2.7), if exists, is unique. Let us find any random process $a^0(\tau) \in \mathbb{C}^n, 0 \le \tau \le T\}$, such that its distribution equals to the measure Q^0 in (2.5).

Theorem 2.3. *Let* (A1) *holds (for some* $T > 0$*). Then* $a^0(\tau)$ *is a unique weak solution of* (2.7)*. So*

$$\mathcal{D}(a^\varepsilon(\tau)) \rightharpoonup \mathcal{D}(a^0(\tau)) \;\; \textit{for} \;\; 0 \le \tau \le T, \quad \textit{as} \;\; \varepsilon \to 0. \tag{2.8}$$

Due to this results the actions $\frac{1}{2}|v_k|^2(\tau)$ of solutions for eq. (2.1) converge in distribution to those of effective equation (2.7).

Assumption (A1) is fulfilled for many equation (2.1). In particular, it holds for any $T > 0$ if vector field P is coercitive in the sense that

$$\langle P(v), v\rangle \le -\alpha_1|v| + \alpha_2 \qquad \forall v \in \mathbb{C}^n, \tag{2.9}$$

for some $\alpha_1 > 0$ and $\alpha_2 \in \mathbb{R}$. This also is a result of Khasminskii; see in [3, Section 9]. Relation (2.9) qualifies the "friction, added to the system" (see the end of Introduction).

3 Properties of convergence (2.8)

We recall that eq. (2.1) is *mixing* if there exists a measure $\mu^\varepsilon \in \mathcal{P}(\mathbb{C}^n)$, called a *stationary measure* for the equation, such that for any $v_0 \in \mathbb{C}^n$

$$\mathcal{D}(v^\varepsilon(\tau; v_0)) \rightharpoonup \mu^\varepsilon \quad \text{as} \quad \tau \to \infty,$$

where $v^\varepsilon(\tau; v_0)$ is a solution of (2.1). Khasminskii proved that if (2.9) holds and $b_k \neq 0$ for all k, then eq. (2.1) is mixing (see [3] for references). In this case eq. (2.7) is mixing as well, see [3].

Theorem 3.1. *Let* (A1) *holds for each* $T > 0$ *and equations* (2.1) *and* (2.7) *are mixing with stationary measures* μ^ε *and* μ^0*. Then*

$$\mu^\varepsilon \rightharpoonup \mu^0 \quad \textit{as} \quad \varepsilon \to 0.$$

Note that this result relates with the effective equation not the limiting as $\varepsilon \to 0$ behaviour of solutions for a-equation (2.4), but that of the original v-equation (2.1). Also note that due to the example of equation (1.2) for the assumption of the theorem to hold, equation (2.1) must contain dissipation.

Let us provide the space of measures $\mathcal{P}(\mathbb{C}^n)$ with any distance *dist* which makes it a complete metric space and defines there a convergence, equivalent to the weak convergence of measures (there are many distances like that). We have:

Theorem 3.2. *Let* (A1) *holds for each* $T > 0$ *and equation* (2.7) *is mixing. Then for any* $v_0 \in \mathbb{C}^n$ *the convergence* (2.8) *is uniform in time:*

$$\lim_{\varepsilon\to 0} \sup_{\tau\geq 0} \operatorname{dist}\left(\mathcal{D}a^\varepsilon(\tau; v_0), \mathcal{D}a^0(\tau; v_0)\right) = 0.$$

Finally we note that close analogies of the three theorem above hold for stochastic PDEs with similar proofs. See [4] and references in that work.

References

[1] V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt. *Mathematical Aspects of Classical and Celestial Mechanics.* Springer, third edition, 2006.

[2] N. N. Bogoliubov and Y. A. Mitropolsky. *Asymptotic Methods in the Theory of Non-linear Oscillations.* Gordon and Breach, New York, 1961.

[3] G. Huang and S. Kuksin. Averaging and mixing for stochastic perturbations of linear conservative systems. `arXiv:2206.00605`, 2022.

[4] G. Huang and S. Kuksin. On averaging and mixing for stochastic PDEs. `arXiv:2203.15622`, 2022.

[5] L. Niederman. Nonlinear stability around an elliptic equilibrium point in a Hamiltonian system. *Nonlinearity*, 11:1465–1479, 1998.