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Valeria Banica

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Institut des hautes études scientifiques
Le Bois-Marie • Route de Chartres
F-91440 BURES-SUR-YVETTE
<http://www.ihes.fr/>

Centre de mathématiques Laurent Schwartz
CMLS, École polytechnique, CNRS, Université
Paris-Saclay
F-91128 PALAISEAU CEDEX
<http://www.math.polytechnique.fr/>

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1-D CUBIC NLS WITH SEVERAL DIRAC MASSES AS INITIAL DATA AND CONSEQUENCES

VALERIA BANICA

ABSTRACT. In this proceedings article we present a result on the 1-D cubic nonlinear Schrödinger equation with a sum of Dirac masses as initial data. We shall give a sketch of the proof. By using this result we show how to construct the evolution in time of a polygonal line through the binormal flow. This equation is a geometric flow for curves in \mathbb{R}^3 and it is used as a model for the evolution of a vortex filament in fluid mechanics. These results were obtained in collaboration with Luis Vega in [4].

In the first section we shall present the 1-D cubic nonlinear Schrödinger equation (NLS) result and sketch its proof in the second section. In the last section we state the binormal flow result and describe the steps of its proof.

1. 1-D CUBIC NLS WITH SEVERAL DIRAC MASSES AS INITIAL DATA

The nonlinear Schrödinger equation with cubic power on \mathbb{R} :

$$i\psi_t + \psi_{xx} \pm |\psi|^2\psi = 0,$$

is known to be well-posed in H^s for $s \geq 0$ ([15], [9]). If $s < 0$ the equation is ill-posed as uniqueness is lost ([27]) and norm-inflation phenomena occur ([11]). We note that the threshold obtained by the rescaled $\lambda\psi(\lambda^2t, \lambda x)$ solutions is $\dot{H}^{-1/2}$. For $s \leq -\frac{1}{2}$ norm inflating phenomena appear with loss of regularity ([8], [24], [33]). For $-\frac{1}{2} < s < 0$ a control of the growth of Sobolev norms of Schwartz solutions has been obtained for the equation posed on the line or the circle ([22], [25]). Besides the Sobolev spaces, well-posedness was proved to hold for data with Fourier transform in L^p spaces, $p < +\infty$ ([39], [17], [10]). Actually a natural space to be considered would be of data with Fourier transform in L^∞ , as this space $\mathcal{F}(L^\infty)$ is also invariant under the above scaling.

We shall focus now on the case of initial data of Dirac mass type. Note that the Dirac mass is borderline for $\dot{H}^{-1/2}$ and that it belongs to $\mathcal{F}(L^\infty)$. For δ_0 as initial data, the 1-D cubic NLS is ill-posed: when looking for a (unique) solution, by using Galilean invariance, one obtains $e^{i \log t} e^{ix^2/4t} / \sqrt{4\pi it}$ and we don't recover the initial data ([27]). A natural change to do is to consider the perturbed cubic 1DNLS

$$i\psi_t + \psi_{xx} \pm \left(|\psi|^2 - \frac{1}{4\pi t} \right) \psi = 0,$$

and get as an explicit solution $e^{ix^2/4t} / \sqrt{4\pi it} = e^{it\Delta} \delta_0(x)$. The problem is however ill-posed, as it was proved in [2] that small smooth perturbations of the solution $e^{ix^2/4t} / \sqrt{4\pi it}$ at time $t = 1$ behave near $t = 0$ as $e^{i \log t} f(x)$ for some regular function f . However, there is a natural geometric way to decide what to choose as a solution after time $t = 0$: it is a solution that

behaves near $t = 0$ as $e^{i \log |t|} \overline{f(-x)} e^{2i \log |2x|}$ ([3]).

We consider now distributions

$$u_0 = \sum_{k \in \mathbb{Z}} \alpha_k \delta_k.$$

Their Fourier transform on \mathbb{R} writes

$$\hat{u}_0(\xi) = \sum_{k \in \mathbb{Z}} \alpha_k e^{-ik\xi},$$

and in particular \hat{u}_0 is 2π -periodic. Imposing $\{\alpha_k\} \in l^{2,s}$:

$$\|\{\alpha_k\}\|_{l^{2,s}}^2 := \sum_{k \in \mathbb{Z}} (1 + |k|)^{2s} |\alpha_k|^2 < \infty,$$

translates into $\hat{u}_0 \in H^s(0, 2\pi)$ and in particular

$$\{\|\hat{u}_0\|_{H^s((2\pi j, 2\pi(j+1)))}\}_{j \in \mathbb{Z}} \in l^\infty.$$

The NLS equation with subcritical nonlinearities $|u|^p u$ with $p < 3$ and such data was solved in [24]. Our first result is the following.

Theorem 1.1 (Solutions of 1-D cubic NLS linked to several Dirac masses as initial data). *Let $s > \frac{1}{2}$, $0 < \gamma < 1$ and $\{\alpha_k\} \in l^{2,s}$. We consider the 1-D cubic NLS equation:*

$$i\partial_t u + \Delta u \pm (|u|^2 - \frac{M}{2\pi t}) u = 0,$$

with $M = \sum_{k \in \mathbb{Z}} |\alpha_k|^2$. There exists $T > 0$ and a unique solution on $(0, T)$ of the form

$$u(t, x) = \sum_{k \in \mathbb{Z}} e^{\mp i \frac{|\alpha_k|^2}{4\pi} \log t} (\alpha_k + R_k(t)) e^{it\Delta} \delta_k(x),$$

with

$$\sup_{0 < t < T} t^{-\gamma} \|\{R_k(t)\}\|_{l^{2,s}} + t \|\{\partial_t R_k(t)\}\|_{l^{2,s}} < C.$$

Moreover, considering as initial data a finite sum of N Dirac masses

$$u_0 = \sum_{k \in \mathbb{Z}} \alpha_k \delta_k,$$

with coefficients of equal modulus $|\alpha_k| = a$, and the equation renormalized with $M = (N - \frac{1}{2})a^2$, we have a unique solution on $(-T, T)$

$$u(t) = e^{it\Delta} u_0 \pm ie^{it\Delta} \int_0^t e^{-i\tau\Delta} \left(\left(|u(\tau)|^2 - \frac{M}{2\pi\tau} \right) u(\tau) \right) d\tau,$$

such that $e^{-it\Delta} \widehat{u}(t) \in C^1((-T, T), H^s(0, 2\pi))$ with

$$\|e^{-it\Delta} u(t) - u(0)\|_{H_{pF}^s} \leq Ct^\gamma, \quad \forall t \in (-T, T).$$

Moreover, if $s \geq 1$ then the solution is global in time.

2. SKETCH OF THE PROOF

We denote $\mathcal{N}(u) = |u|^2 u$. Plugging $u(t) = \sum_{k \in \mathbb{Z}} A_k(t) e^{it\Delta} \delta_k$ into the equation we get

$$\sum_{k \in \mathbb{Z}} i \partial_t A_k(t) e^{it\Delta} \delta_k = \mp \mathcal{N} \left(\sum_{j \in \mathbb{Z}} A_j(t) e^{it\Delta} \delta_j \right) \pm \frac{M}{2\pi t} \left(\sum_{k \in \mathbb{Z}} A_k(t) e^{it\Delta} \delta_k \right).$$

The family $e^{it\Delta} \delta_k(x) = e^{i(x-k)^2/4t} / \sqrt{4\pi it}$ is an orthonormal family of $L^2(0, 4\pi t)$, so by taking the scalar product of $L^2(0, 4\pi t)$ with $e^{it\Delta} \delta_k$ we obtain

$$i \partial_t A_k(t) = \mp \int_0^{4\pi t} \mathcal{N} \left(\sum_{j \in \mathbb{Z}} A_j(t) \frac{e^{i \frac{(x-j)^2}{4t}}}{\sqrt{4\pi it}} \right) \frac{-e^{i \frac{(x-k)^2}{4t}}}{\sqrt{4\pi it}} dx \pm \frac{M}{2\pi t} A_k(t).$$

Note that $\{A_j\} \in l^{2,s} \subset l^1$ and we develop the cubic power to get the discrete system

$$i \partial_t A_k(t) = \mp \frac{1}{4\pi t} \sum_{k-j_1+j_2-j_3=0} e^{-i \frac{k^2-j_1^2+j_2^2-j_3^2}{4t}} A_{j_1}(t) \overline{A_{j_2}(t)} A_{j_3}(t) \pm \frac{M}{2\pi t} A_k(t).$$

This system conserves the mass:

$$\begin{aligned} \partial_t \sum_k |A_k(t)|^2 &= \mp \frac{1}{4\pi t} \Im \sum_{k-j_1+j_2-j_3=0} e^{-i \frac{k^2-j_1^2+j_2^2-j_3^2}{4t}} A_{j_1}(t) \overline{A_{j_2}(t)} A_{j_3}(t) \overline{A_k(t)} \\ &= \mp \frac{1}{8\pi t i} \left(\sum_{k-j_1+j_2-j_3=0} e^{-i \frac{k^2-j_1^2+j_2^2-j_3^2}{4t}} A_{j_1}(t) \overline{A_{j_2}(t)} A_{j_3}(t) \overline{A_k(t)} \right. \\ &\quad \left. - \sum_{j_3-j_2+j_1-k=0} e^{-i \frac{j_3^2-j_2^2+j_1^2-k^2}{4t}} A_{j_2}(t) \overline{A_{j_1}(t)} A_k(t) \overline{A_{j_3}(t)} \right) = 0. \end{aligned}$$

We split the summation indices into the following two sets:

$$NR_k = \{(j_1, j_2, j_3) \in \mathbb{Z}^3, k - j_1 + j_2 - j_3 = 0, k^2 - j_1^2 + j_2^2 - j_3^2 \neq 0\},$$

$$Res_k = \{(j_1, j_2, j_3) \in \mathbb{Z}^3, k - j_1 + j_2 - j_3 = 0, k^2 - j_1^2 + j_2^2 - j_3^2 = 0\}.$$

As we are in one dimension, the second set is simply

$$Res_k = \{(k, j, j), (j, j, k), j \in \mathbb{Z}\}.$$

Finally the system writes

$$\begin{aligned} i \partial_t A_k(t) &= \mp \sum_{(j_1, j_2, j_3) \in NR_k} \frac{e^{-i \frac{k^2-j_1^2+j_2^2-j_3^2}{4t}}}{4\pi t} A_{j_1}(t) \overline{A_{j_2}(t)} A_{j_3}(t) \\ &\quad \mp \frac{A_k(t)}{4\pi t} (2 \sum_j |A_j(t)|^2 - |A_k(t)|^2 - 2M). \end{aligned}$$

As $M = \sum_j |\alpha_j|^2$, finding a solution for $t > 0$ satisfying

$$\lim_{t \rightarrow 0} |A_j(t)| = |\alpha_j|,$$

is equivalent to finding a solution for $t > 0$ satisfying also this limit, for the following also mass-conserving system:

$$i\partial_t A_k(t) = \mp \sum_{(j_1, j_2, j_3) \in NR_k} \frac{e^{-i\frac{k^2 - j_1^2 + j_2^2 - j_3^2}{4t}}}{4\pi t} A_{j_1}(t) \overline{A_{j_2}(t)} A_{j_3}(t) \pm \frac{1}{4\pi t} |A_k(t)|^2 A_k(t).$$

By doing a change of phase $A_k(t) = e^{i\frac{|\alpha_k|^2}{4\pi} \log t} \tilde{A}_k(t)$ we get as a system

$$i\partial_t \tilde{A}_k(t) = \mp f_k(t) \pm \frac{1}{4\pi t} (|\tilde{A}_k(t)|^2 - |\alpha_k|^2) \tilde{A}_k(t),$$

where

$$f_k(t) = \frac{1}{4\pi t} \sum_{(j_1, j_2, j_3) \in NR_k} e^{-i\frac{k^2 - j_1^2 + j_2^2 - j_3^2}{4t}} e^{-i\frac{|\alpha_k|^2 - |\alpha_{j_1}|^2 + |\alpha_{j_2}|^2 - |\alpha_{j_3}|^2}{4\pi} \log t} \tilde{A}_{j_1}(t) \overline{\tilde{A}_{j_2}(t)} \tilde{A}_{j_3}(t).$$

As a solution of this system satisfies

$$\partial_t |A_k(t)|^2 = \mp 2\Im(f_k(t) \overline{\tilde{A}_k(t)}),$$

obtaining a solution for $t > 0$ with

$$\lim_{t \rightarrow 0} |\tilde{A}_k(t)| = |\alpha_k|,$$

is equivalent to obtaining a solution for $t > 0$ also satisfying the limit conditions, for the following system, that also enjoys the above mass evolution:

$$i\partial_t \tilde{A}_k(t) = \mp f_k(t) - \frac{1}{2\pi t} \int_0^t \Im(f_k(\tau) \overline{\tilde{A}_k(\tau)}) d\tau \tilde{A}_k(t).$$

Recall that we want to obtain the existence of $A_k(t) = e^{\mp i\frac{|\alpha_k|^2}{4\pi} \log t} (\alpha_k + R_k(t))$, with

$$\begin{aligned} \{R_k\} \in X^\gamma &:= \{\{f_k\} \in C^1((0, T); l^{2,s}), \|\{f_k\}\|_{X^\gamma} \\ &:= \sup_{0 < t < T} t^{-\gamma} \|\{f_k(t)\}\|_{l^{2,s}} + t \|\{\partial_t f_k(t)\}\|_{l^{2,s}} < \infty\}. \end{aligned}$$

We prove this by a fixed point argument for the operator $\Phi : \{R_k\} \rightarrow \{\Phi_k(\{R_j\})\}$ defined as

$$\Phi_k(\{R_j\})(t) := \mp i \int_0^t g_k(\tau) d\tau - i \int_0^t \int_0^\tau \Im(g_k(s) \overline{(\alpha_k + R_k(s))}) ds (\alpha_k + R_k(\tau)) \frac{d\tau}{4\pi\tau},$$

where

$$\begin{aligned} g_k(t) &= \frac{1}{4\pi t} \sum_{(j_1, j_2, j_3) \in NR_k} e^{-i\frac{k^2 - j_1^2 + j_2^2 - j_3^2}{4t}} e^{-i\frac{|\alpha_k|^2 - |\alpha_{j_1}|^2 + |\alpha_{j_2}|^2 - |\alpha_{j_3}|^2}{4\pi} \log t} \\ &\quad \times (\alpha_{j_1} + R_{j_1}(t)) \overline{(\alpha_{j_2} + R_{j_2}(t))} (\alpha_{j_3} + R_{j_3}(t)), \end{aligned}$$

in a ball of X^γ of radius δ , chosen in terms of $\|\{\alpha_j\}\|_{l^{2,s}}$, and T small with respect $\|\{\alpha_j\}\|_{l^{2,s}}$ and γ .

For getting bounds on the operator Φ we perform integrations by parts to get integrability in time. We consider here for instance the free term:

$$\begin{aligned} I_k(t) &:= i \int_0^t \sum_{(j_1, j_2, j_3) \in NR_k} \frac{e^{-i \frac{k^2 - j_1^2 + j_2^2 - j_3^2}{4\tau}} e^{-i \frac{|\alpha_k|^2 - |\alpha_{j_1}|^2 + |\alpha_{j_2}|^2 - |\alpha_{j_3}|^2}{4\pi} \log t} \alpha_{j_1} \overline{\alpha_{j_2}} \alpha_{j_3}}{4\pi\tau} d\tau \\ &= t \sum_{(j_1, j_2, j_3) \in NR_k} \frac{e^{-i \frac{k^2 - j_1^2 + j_2^2 - j_3^2}{4t}} e^{-i \frac{|\alpha_k|^2 - |\alpha_{j_1}|^2 + |\alpha_{j_2}|^2 - |\alpha_{j_3}|^2}{4\pi} \log t} \alpha_{j_1} \overline{\alpha_{j_2}} \alpha_{j_3}}{\pi(k^2 - j_1^2 + j_2^2 - j_3^2)} \\ &\quad - \int_0^t \sum_{(j_1, j_2, j_3) \in NR_k} \frac{e^{-i \frac{k^2 - j_1^2 + j_2^2 - j_3^2}{4\tau}} e^{-i \frac{|\alpha_k|^2 - |\alpha_{j_1}|^2 + |\alpha_{j_2}|^2 - |\alpha_{j_3}|^2}{4\pi} \log \tau} \alpha_{j_1} \overline{\alpha_{j_2}} \alpha_{j_3}}{\pi(k^2 - j_1^2 + j_2^2 - j_3^2)} d\tau. \end{aligned}$$

On the non-resonant set $1 \leq |k^2 - j_1^2 + j_2^2 - j_3^2|$, so

$$\|I_k(t)\|_{l^2, s} \leq Ct \|\{\alpha_j\}\|_{l^2, s}^3,$$

where we used the following weighted Young discrete inequality:

$$\|\{M_j\} \star \{N_j\} \star \{P_j\}\|_{l^2, s} \leq C \|\{M_j\}\|_{l^2, s} \|\{N_j\}\|_{l^2, s} \|\{P_j\}\|_{l^2, s}.$$

The other terms can be estimated similarly. Moreover, the regularity imposed on $\{\alpha_k\}$ can be lowered by taking in account in the discrete summations the decay $|k^2 - j_1^2 + j_2^2 - j_3^2|^{-1}$. Finally, for $s > 1$ the extension in time is obtained by getting a control on the growth of $\|\{\alpha_j + R_j(t)\}\|_{L^\infty(0, T)l^{2,1}}$.

3. EVOLUTION OF POLYGONAL LINES THROUGH THE BINORMAL FLOW

A vortex filament in 3-D fluids appears when vorticity is large and concentrated in a thin tube around a curve $\chi(t)$ in \mathbb{R}^3 . A classical model for the dynamics of one vortex filament is the binormal flow (BF), named also LIA from Local Induction Approximation or VFE from Vortex Filament equation. It is a geometric flow of curves $\chi(t)$ in \mathbb{R}^3 described by

$$\chi_t = \chi_x \wedge \chi_{xx}.$$

Here x stands for the arclength parameter of the curve $\chi(t)$. This was derived as formal asymptotics by L. da Rios in 1906 in its PhD advised by Levi-Civita ([12]), by Arms and Hama in 1965 ([1]), and also in Navier-Stokes matched asymptotics ([7], see also [31],[36],[35]). The only rigorous result is [20]: with less hypothesis on the persistence of concentration of vorticity, BF is derived rigorously, moreover for curves not necessarily smooth.

If one considers the filament function

$$\psi(t, x) = c(t, x) e^{i \int_0^x \tau(t, s) ds},$$

it is easy to check that it satisfies the NLS equation

$$i\psi_t + \psi_{xx} + \frac{1}{2} (|\psi|^2 - A(t)) \psi = 0,$$

where $A(t)$ is in terms of the curvature and torsion $(c, \tau)(t, 0)$. This important remark has been made by Hasimoto in 1972 and it has allowed the transfer of informations from NLS to BF ([19]). The Hasimoto transform can be seen as an inverse of the Madelung transform that

connects the Gross-Pitaevskii equation to Euler equation with quantum pressure. Actually also here, the system satisfied by the curvature and torsion is a Euler-Korteweg one.

In order to avoid issues related to vanishing curvature, Bishop parallel frames ([5]) were used as follows in [26]. Let (T, e_1, e_2) be another frame than the Frenet frame (T, n, b) , governed by

$$\begin{pmatrix} T \\ e_1 \\ e_2 \end{pmatrix}_x = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & 0 \\ -\beta & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ e_1 \\ e_2 \end{pmatrix}.$$

If we denote $N = e_1 + ie_2$ and if $T(t)$ is the tangent vector of a curve $\chi(t)$ solution of BF, then one can compute

$$T_x = \Re(\bar{\psi}N), N_x = -\psi T, T_t = \Im\bar{\psi}_x N, \quad N_t = -i\psi_x T + i(|\psi|^2 - A(t))N,$$

and verify that

$$\psi(t, x) = \alpha(t, x) + i\beta(t, x)$$

is a NLS solution with $A(t) = \alpha^2(t, 0) + \beta^2(t, 0)$. Such a frame can be obtained by a rotation of the Frenet frame, $N(t, x) = (n + ib)(t, x)e^{i\int_0^x \tau(t,s)ds}$.

Conversely, given a NLS solution ψ , (e_0, e_1, e_2) an orthonormal basis of \mathbb{R}^3 and P a point of \mathbb{R}^3 , one might construct a solution of BF in the following way. First, $(T, N)(t, x)$ can be constructed by imposing $(T, N)(t_0, x_0) = (e_0, e_1 + ie_2)$ and the above evolutions laws. Then $\chi(t, x)$ defined as

$$\chi(t, x) = P + \int_{t_0}^t (T \wedge T_x)(\tau, x_0) d\tau + \int_{x_0}^x T(t, s) ds,$$

is a solution of BF, as T solves the Schrödinger map $T_t = T \wedge T_{xx} = (T \wedge T_x)_x$. Summarizing, this recipe can be used to construct solutions of BF starting from solutions of NLS. However, recovering the geometric properties of the solution of BF is not obvious at all.

Existence results for BF for regular curves have been obtained by various methods [12], [19], [32], [38]. Recently the less regular case of currents has been considered in a weak formulation of the equation [21].

We shall focus now on curves developing a corner in finite time. We start with some facts on the self-similar solutions of BF, that is solutions of the type

$$\chi(t, x) = \sqrt{t}G\left(\frac{x}{\sqrt{t}}\right).$$

We recall that since the 70's the BF and its self-similar solutions were considered in works on vortex dynamics in superfluids [37, 6, 30], in ferromagnetism [29, 28], in aortic heart valve leaflet miocardic modeling [34], [40].

In [29] it was shown that self-similar solutions form a family $\{\chi_a\}_{a \in \mathbb{R}^{+*}}$ characterized by the explicit curvature and torsion $(c_a, \tau_a)(t, x) = \left(\frac{a}{\sqrt{t}}, \frac{x}{2t}\right)$. Numerical computations on this formation of a singularity in finite time were given in [6]. In [18] this was proved rigorously. More precisely, it was shown that a corner appear at time $t = 0$:

$$|\chi_a(t, x) - x(A_a^+ \mathbb{I}_{[0, \infty)}(x) + A_a^- \mathbb{I}_{(-\infty, 0]}(x))| \leq 2a\sqrt{t},$$

with $A_a^\pm \in \mathbb{S}^2$ distincts, non-opposite and

$$\sin \frac{\widehat{(A_a^+, -A_a^-)}}{2} = e^{-a^2/2}.$$

In particular, any corner can be obtained in finite time from a rotated and translated self-similar solution.

The phenomenon of the formation and instantaneous disappearance of one corner has been well understood through the self-similar solutions, and through the study in [3] of the evolution of non-closed curves with one corner and curvature in weighted space. On the other hand, a planar regular polygon with M sides is expected to evolve through the binormal flow to skew polygons with Mq sides at times of type p/q (numerical simulations [16], [21] and integration of the Frenet frame at rational times [13]).

In the present paper we place ourselves in the framework of initial data being polygonal lines. The results presented below are an important step forward to fill the gap between the case of one corner and the much more delicate issue of the polygon.

Theorem 3.1 (Evolution of polygonal lines through the binormal flow). *Let χ_0 be an ar-length parametrized polygonal line with corners located at $x \in \mathbb{Z}$, with the sequence of angles $\theta_n \in (0, \pi)$ such that the sequence defined by*

$$\sqrt{-\frac{2}{\pi} \log(\sin(\theta_n/2))},$$

belongs to $l^{2,3}$. Then there exists $\chi(t)$, smooth solution of the binormal flow on \mathbb{R}^ and solution in the weak sense on \mathbb{R} , with*

$$|\chi(t, x) - \chi_0(x)| \leq C\sqrt{t}, \quad \forall x \in \mathbb{R}, |t| \leq 1.$$

Remark 3.2. *Under suitable conditions on the initial data χ_0 , we show that the evolution can have an intermittent behaviour: at times $t_{p,q} = \frac{1}{2\pi} \frac{p}{q}$ the curvature of $\chi(t)$ displays concentrations near the locations x such that $x \in \frac{1}{q}\mathbb{Z}$, and $\chi(t)$ is almost a straight segment in between. We prove this result by displaying a Talbot effect for some solutions of Theorem 1.1.*

The main steps of the proof of Theorem 3.1 are the following:

- we define $\alpha_k = \tilde{\alpha}_k e^{i\gamma_k}$, with γ_k designed in terms of the curvature angles and torsion angles of the polygonal line in some specific way that arises at the end of the proof,
- via Theorem 1.1 get a NLS solution $u(t)$, smooth for $t > 0$,
- consider the BF solution $\chi(t)$ obtained from $u(t)$,
- get a trace $\chi(0)$ for $\chi(t)$ as t goes to zero; the goal is now to show that it is χ_0 ,
- for $x \in \mathbb{R}$ obtain a limit as t goes to zero for $T(t, x)$, with a self-similar decay $\frac{\sqrt{t}}{d(x, \mathbb{Z})}$ for $x \notin \mathbb{Z}$,
- prove that the vectors $T(0, x)$ are constant for $k < x < k + 1$, so $\chi(0)$ is a polygonal line,
- recover a self-similar BF profile $\chi_{|\alpha_k|}$ on self-similar paths: for instance $\exists t_n \xrightarrow{n \rightarrow \infty} 0, \exists \Theta_k$ rotation s.t.

$$\lim_{n \rightarrow \infty} T(t_n, k + x\sqrt{t_n}) = \Theta_k(T_{|\alpha_k|}(x)),$$

- recover the curvature angles of χ_0 by the following successive links, for $|x|$ and n large enough:

$$T(0, k^\pm) \rightsquigarrow T(0, k + x\sqrt{t_n}) \rightsquigarrow T(t_n, k + x\sqrt{t_n}) \rightsquigarrow \Theta_k A_{|\alpha_k|}^\pm,$$

- recover the torsion angles of χ_0 by using also a similar analysis for modulated normal vectors $\tilde{N}(t, x) = e^{i \sum_j |\alpha_j|^2 \log \frac{x-j}{\sqrt{t}}} N(t, x)$,
- recover χ_0 after a translation and a rotation,
- extend to negative time by using the time reversibility of BF so by solving BF for positive times with data $\chi_0(-s)$, which is also a polygonal line that enters the framework above.

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(V. Banica) LABORATOIRE JACQUES-LOUIS LIONS (UMR 7598) B.C. 187, 4 PLACE JUSSIEU 75005 PARIS, FRANCE

E-mail address: Valeria.Banica@ljl11.math.upmc.fr