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THE CUBIC SZEGŐ FLOW AT LOW REGULARITY

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ABSTRACT. We prove that the cubic Szegő equation is well posed on the space BMO_+ of functions of bounded mean oscillation in the Hardy class of the disc, and we establish the Hölder regularity of this flow in the L^2 distance. We also show that the Cauchy problem is illposed on the corresponding L^∞ space.

1. INTRODUCTION

This paper is devoted to low regularity solutions of the cubic Szegő equation on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$,

$$(1) \quad i\partial_t u = \Pi(|u|^2 u)$$

where $\Pi : L^2(\mathbb{T}) \rightarrow L^2_+(\mathbb{T})$ denotes the orthogonal projector onto the closed subspace $L^2_+(\mathbb{T})$ of $L^2(\mathbb{T})$ defined by the cancellation of all negative Fourier modes,

$$\forall k < 0, \quad \widehat{u}(k) = 0.$$

Recall that $L^2_+(\mathbb{T})$ can be identified to the Hardy space $\mathbb{H}^2(\mathbb{D})$ consisting of holomorphic functions u on the unit disc such that

$$\sup_{r < 1} \int_0^{2\pi} |u(re^{ix})|^2 dx < \infty.$$

In the sequel, we shall make use of this identification freely.

Equation (1) was introduced by S. Grellier and the first author in [5], where a flow on $H^s_+(\mathbb{T}) := H^s(\mathbb{T}) \cap L^2_+(\mathbb{T})$, $s \geq 1/2$, was defined, and where a Lax pair structure was discovered. In [8], this equation was identified as the time averaged effective system to the half wave equation on \mathbb{T} . In [6], more precise integrability properties were established, while in [7] an explicit formula for H^s solutions was derived. Finally, a general nonlinear Fourier transform was constructed in [9], where almost periodicity of solutions in $H^{1/2}_+$ and growth of higher Sobolev

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norms were proved. Furthermore, analyticity of solutions was studied in [10].

Since Π is a pseudodifferential operator of order 0, it is natural to ask about solving Equation (1) for initial data with low regularity. For instance, the ordinary differential equation

$$(2) \quad i\partial_t u = |u|^2 u$$

is wellposed on $L^\infty(\mathbb{T})$, with the explicit formula

$$u(t, x) = e^{-it|u(0,x)|^2} u(0, x) .$$

The purpose of this paper is to investigate how this property is modified by the action of the pseudodifferential operator Π . It is well known that Π is not bounded on $L^\infty(\mathbb{T})$. The space

$$\text{BMO}_+(\mathbb{T}) = \{\Pi(b), b \in L^\infty(\mathbb{T})\}$$

was identified by Fefferman [3] as the intersection of $L^2_+(\mathbb{T})$ with the space $\text{BMO}(\mathbb{T})$ of functions of bounded mean oscillation introduced by John and Nirenberg, see [13], [4], as the space of functions $f \in L^1(\mathbb{T})$ such that

$$(3) \quad \sup_I \frac{1}{|I|} \int_I |f(x) - \langle f \rangle_I| dx < +\infty, \quad \langle f \rangle_I := \frac{1}{|I|} \int_I f(x) dx ,$$

where the supremum above is taken on all intervals $I \subset \mathbb{T}$. The space BMO_+ is also the dual of

$$L^1_+(\mathbb{T}) = \{h \in L^1(\mathbb{T}) : \forall k < 0, \widehat{h}(k) = 0\} .$$

For every $u \in \text{BMO}_+(\mathbb{T})$, we set

$$\|u\|_{\text{BMO}} = \inf\{\|b\|_{L^\infty}, b \in L^\infty(\mathbb{T}), \Pi(b) = u\} = \|u\|_{(L^1_+)'}$$

Our main result is the following.

Theorem 1. *For every $u_0 \in \text{BMO}_+(\mathbb{T})$, there exists a unique function $u \in C^1(\mathbb{R}, L^2_+(\mathbb{T})) \cap C_{w*}(\mathbb{R}, \text{BMO}_+(\mathbb{T}))$, solution of the initial value problem*

$$(4) \quad i\partial_t u = \Pi(|u|^2 u), \quad u(0) = u_0 .$$

Furthermore, $\|u(t)\|_{\text{BMO}} = \|u_0\|_{\text{BMO}}$. Moreover, if u, v are two BMO solutions of (1) satisfying

$$\|u(0)\|_{\text{BMO}} + \|v(0)\|_{\text{BMO}} \leq M ,$$

there exists a constant K , depending only on M , such that, for every $t \in \mathbb{R}$,

$$(5) \quad \|u(t) - v(t)\|_{L^2} \leq K \|u(0) - v(0)\|_{L^2}^{\alpha(t)}, \quad \alpha(t) := e^{-K|t|} .$$

Next we come to propagation of Sobolev regularity. In the low regularity case, it is only partially obtained, as a consequence of the stability estimate (5).

Corollary 1. *Let u be a BMO solution of the cubic Szegő equation, as given by Theorem 1. Assume $u(0) = u_0 \in H^s$ for some $s > 0$. Then, if $s \geq 1/2$, $u(t) \in H^s(\mathbb{T})$ for every $t \in \mathbb{R}$. In the case $0 < s < 1/2$, there exists $K > 0$, depending only on a bound of $\|u_0\|_{\text{BMO}}$, such that*

$$\forall t \in \mathbb{R}, u(t) \in H^{s(t)}(\mathbb{T}), \quad s(t) := \frac{se^{-K|t|}}{1 - 2s + 2se^{-K|t|}}.$$

Remark 1.

- We do not know whether the above exponent $s(t)$ is optimal or not. If it is optimal, such a loss of regularity could be compared to the one established by Bahouri and Chemin in Theorem 1.3 of [1] for the bidimensional incompressible Euler flow with bounded vorticity.
- The above corollary has a local version, which will be established in the forthcoming paper [11].

In the beginning of this note, we have seen that the ordinary differential equation (2) is well posed on $L^\infty(\mathbb{T})$. In contrast, using the John–Nirenberg definition (3), it is easy to prove that this equation is not wellposed on $\text{BMO}(\mathbb{T})$. Indeed, though $u_0(x) = \log |\sin x|$ belongs to $\text{BMO}(\mathbb{T})$, one can check that, for every $t \neq 0$, the function

$$u(t, x) = (\log |\sin x|)e^{-it(\log |\sin x|)^2}$$

does not belong to $\text{BMO}(\mathbb{T})$, because its average on $[\varepsilon, 2\varepsilon]$ is bounded as ε tends to 0. Somewhat symmetrically, the next result shows that the Szegő equation is illposed on L^∞ . We denote by $C_+(\mathbb{T}) = C(\mathbb{T}) \cap L_+^2(\mathbb{T})$ the Banach space of continuous functions on the circle with nonnegative Fourier modes.

Theorem 2. *There exists a dense G_δ subset \mathcal{G} of $C_+(\mathbb{T})$ such that, for every $u_0 \in \mathcal{G}$, the solution u of (4) satisfies*

$$\forall T > 0, u \notin L^\infty([0, T] \times \mathbb{T}).$$

The present note will give a sketch of the proof of Theorem 1, Corollary 1 and Theorem 2. An extended version with more detailed proofs and additional results is in preparation [11].

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2. PROOF OF THEOREM 1

The proof of Theorem 1 is based on two arguments. The first one is a characterization of $\text{BMO}_+(\mathbb{T})$ which was established by Nehari [16] before the John–Nirenberg paper. Nehari’s result — see also Theorem 1.1 of Peller [17] — claims that, given $u \in L_+^2(\mathbb{T})$, the Hankel operator $\Gamma_{\hat{u}}$ defined on finitely supported sequence $\mathbf{x} := (x_n)_{n \geq 0}$ by

$$[\Gamma_{\hat{u}}(\mathbf{x})]_p = \sum_{n=0}^{\infty} \hat{u}(p+n)x_n$$

extends as a bounded operator on $\ell^2(\mathbb{N})$ if and only if $u \in \text{BMO}_+(\mathbb{T})$, and that

$$\|\Gamma_{\widehat{u}}\|_{\ell^2 \rightarrow \ell^2} = \|u\|_{\text{BMO}} .$$

As we will recall below, it turns out that the Lax pair discovered in [5] allows to prove that, if a u is a smooth solution of (1), the operator norm $\|\Gamma_{\widehat{u}(t)}\|_{\ell^2 \rightarrow \ell^2}$ is independent of t . This provides a BMO bound for the sequence (u_n) of smooth solutions of (1) which approximates u_0 at $t = 0$ in $\mathcal{B}_{\text{BMO}}(\|u_0\|_{\text{BMO}})$.

The second argument relies on the John–Nirenberg inequality [13], [4], which claims that $\text{BMO}_+(\mathbb{T}) \subset L^p(\mathbb{T})$ for every $p < \infty$, and that there exists a universal constant $C > 0$ such that, for every $v \in \text{BMO}_+(\mathbb{T})$, for every $p \in [1, \infty)$,

$$\|u\|_{L^p} \leq C p \|v\|_{\text{BMO}} .$$

This inequality will allow us to prove that the sequence (u_n) is a Cauchy sequence in $C([-T, T], L^2_+(\mathbb{T}))$ for every $T < \infty$, leading to existence of solution u .

Let us come to the detailed proof of Theorem 1. We first recall the Lax pair structure of the cubic Szegő equation, as established in [5] and revisited in [7]. For every $u \in \text{BMO}_+(\mathbb{T})$, define the antilinear Hankel operator

$$H_u : L^2_+(\mathbb{T}) \rightarrow L^2_+(\mathbb{T})$$

by the formula

$$H_u(h) = \Pi(u\bar{h}) , \quad h \in L^2_+(\mathbb{T}) .$$

It is easy to check that H_u is bounded on $L^2_+(\mathbb{T})$, and that

$$\widehat{H_u(h)} = \Gamma_{\widehat{u}}(\widehat{h}) , \quad \langle H_u(h_1), h_2 \rangle = \langle H_u(h_2), h_1 \rangle ,$$

where $\langle f, g \rangle$ denotes the usual L^2 inner product. In particular,

$$H_u^2 \simeq \Gamma_{\widehat{u}} \Gamma_{\widehat{u}}^*$$

is a linear positive selfadjoint operator. From Nehari's theorem, we have

$$(6) \quad \|H_u\|_{L^2_+ \rightarrow L^2_+} = \|u\|_{\text{BMO}} .$$

Next we claim that, for every $a, b, c \in L^\infty(\mathbb{T})$,

$$(7) \quad H_{\Pi(a\bar{b}c)} = T_{a\bar{b}}H_c + H_a T_{b\bar{c}} - H_a H_b H_c ,$$

where, for every $m \in L^\infty(\mathbb{T})$, the Toeplitz operator T_m is defined by

$$T_m(h) = \Pi(mh) , \quad h \in L^2_+(\mathbb{T}) .$$

Indeed, given $h \in L_+^2(\mathbb{T})$, we have

$$\begin{aligned} H_{\Pi(\bar{a}\bar{b}c)}(h) &= \Pi(\Pi(\bar{a}\bar{b}c)\bar{h}) = \Pi(\bar{a}\bar{b}c\bar{h}) \\ &= \Pi(\bar{a}\bar{b}\Pi(c\bar{h})) + \Pi(\bar{a}\bar{b}(I - \Pi)(c\bar{h})) \\ &= T_{\bar{a}\bar{b}}H_c(h) + H_a \left(\overline{b(I - \Pi)(c\bar{h})} \right) . \end{aligned}$$

The proof of (7) is completed by observing that

$$\overline{b(I - \Pi)(c\bar{h})} = \Pi \left(\overline{b(I - \Pi)(c\bar{h})} \right) = T_{\bar{b}\bar{c}}(h) - H_b H_c(h) .$$

Now assume that u is a smooth solution to (1). Combining the equation and identity (7), we have

$$\frac{d}{dt}H_u = -iH_{\Pi(|u|^2u)} = -i(H_u T_{|u|^2} + T_{|u|^2}H_u - H_u^3) = [B_u, H_u] ,$$

where $[B, C]$ denotes the commutator of the operators B, C ,

$$B_u := -iT_{|u|^2} + \frac{i}{2}H_u^2 ,$$

and where we have used the antilinearity of H_u in writing

$$i(H_u A + A H_u) = [iA, H_u]$$

for every linear operator A . Notice that B_u is an antiselfadjoint linear operator on $L_+^2(\mathbb{T})$. Solving the linear ODE

$$(8) \quad \frac{dU}{dt} = B_u U , \quad U(0) = I .$$

in the space of bounded operators on L_+^2 , we get a one parameter family $U(t)$ of unitary operators, which satisfies

$$(9) \quad \forall t \in \mathbb{R} , \quad H_{u(t)} = U(t)H_{u(0)}U(t)^* .$$

From (9) and (6), we conclude

$$(10) \quad \forall t \in \mathbb{R} , \quad \|u(t)\|_{\text{BMO}} = \|u_0\|_{\text{BMO}} .$$

We now come to the second step of the proof, for which the main point is the following stability lemma.

Lemma 1. *Let u, v be two smooth solutions of (1), satisfying*

$$\|u_0\|_{\text{BMO}} + \|v_0\|_{\text{BMO}} \leq M .$$

There exists a constant K , depending only on M , such that, for every $t \in \mathbb{R}$,

$$\|u(t) - v(t)\|_{L^2} \leq K \|u_0 - v_0\|_{L^2}^{e^{-K|t|}} .$$

Proof. Recall that we denote by

$$\langle f, g \rangle = \int_0^{2\pi} f(e^{ix}) \overline{g(e^{ix})} \frac{dx}{2\pi}$$

the inner product on $L^2(\mathbb{T})$. Set $N(t) := \|u(t) - v(t)\|_{L^2}^2$. Assume $t \geq 0$ for simplicity, and compute

$$\frac{dN}{dt} = 2\operatorname{Im} \langle \Pi(|u|^2 u) - \Pi(|v|^2 v), u - v \rangle .$$

Applying the Taylor formula, we have, introducing $w_\theta := \theta u + (1 - \theta)v$ for $\theta \in [0, 1]$,

$$\Pi(|u|^2 u) - \Pi(|v|^2 v) = \int_0^1 (2T_{|w_\theta|^2} + H_{w_\theta^2})(u - v) d\theta .$$

Since $T_{|w_\theta|^2}$ is selfadjoint, its contribution to the imaginary part of the inner product cancels, and we are left with

$$\frac{dN}{dt} = 2 \int_0^1 \operatorname{Im} \langle H_{w_\theta^2}(u - v), u - v \rangle d\theta .$$

Using identity (7) with $a = c = w_\theta$ and $b = 1$, we obtain

$$\begin{aligned} \frac{dN}{dt} &= 2 \int_0^1 \operatorname{Im} \langle (T_{w_\theta} H_{w_\theta} + H_{w_\theta} T_{\bar{w}_\theta} - H_{w_\theta} H_1 H_{w_\theta})(u - v), u - v \rangle d\theta \\ &= 4 \int_0^1 \operatorname{Im} \langle H_{w_\theta}(u - v), \bar{w}_\theta(u - v) \rangle d\theta + 2 \int_0^1 \operatorname{Im} (\langle w_\theta, u - v \rangle^2) d\theta . \end{aligned}$$

From the conservation of the BMO norm (10), we already know that $\|w_\theta\|_{\text{BMO}} \leq M$, and thus

$$\|H_{w_\theta}\|_{L^2_+ \rightarrow L^2_+} \leq M , \quad \|w_\theta\|_{L^p} \leq C M p .$$

Using Hölder's inequality, we infer, for large p and for every time $t \geq 0$,

$$\begin{aligned} |\langle H_{w_\theta}(u - v), \bar{w}_\theta(u - v) \rangle| &\leq M \|u - v\|_{L^2} \|w_\theta |u - v|^{2/p} |u - v|^{1-2/p}\|_{L^2} \\ &\leq M \|u - v\|_{L^2} \|w_\theta |u - v|^{2/p}\|_{L^p} \| |u - v|^{1-2/p} \|_{L^{2p/p-2}} \\ &\leq M (C M p)^{1+2/p} \|u - v\|_{L^2}^{2-2/p} \\ &\leq \tilde{C}(M) p N^{1-1/p} . \end{aligned}$$

We now choose, at a given time $t \geq 0$,

$$p = p(t) = 2 + \log(M^2/N(t)) \geq 2 ,$$

since, by the conservation of L^2 norms of u and v ,

$$N(t) \leq (\|u_0\|_{L^2} + \|v_0\|_{L^2})^2 \leq M^2 .$$

We infer

$$\left| \frac{dN}{dt} \right| \leq K(M) N (2 + \log(M^2/N)) .$$

Solving this differential inequality, we obtain the lemma. \square

Let us complete the proof of Theorem 1. Let $u_0 \in \text{BMO}_+(\mathbb{T})$. Select a sequence (u_0^n) of smooth functions in L_+^2 such that

$$\|u_0^n - u_0\|_{L^2} \rightarrow 0, \quad \limsup \|u_0^n\|_{\text{BMO}} \leq \|u_0\|_{\text{BMO}}.$$

For instance, one can choose

$$u_0^n(e^{ix}) = u_0(r_n e^{ix}),$$

where r_n is any sequence of positive numbers smaller than 1 converging to 1. Denote by u^n the solution of (1) with initial datum u_0^n . Then Lemma 1 implies that (u^n) is a Cauchy sequence in $C([-T, T], L_+^2)$ for every $T > 0$, hence it converges to $u \in C(\mathbb{R}, L_+^2)$. Furthermore,

$$\|u^n(t)\|_{\text{BMO}} = \|u_0^n\|_{\text{BMO}},$$

hence $u_n(t) \rightarrow u(t)$ in L^p for every $p < \infty$, locally uniformly in time. This allows to pass to the limit in Equation (1), so that u is a solution of (4), and moreover

$$\|u(t)\|_{\text{BMO}} \leq \limsup \|u_0^n\|_{\text{BMO}} \leq \|u_0\|_{\text{BMO}}.$$

It remains to prove uniqueness of such solutions, and the conservation of the BMO norm. For uniqueness, we observe that the proof of Lemma 1 can be easily extended to solutions $u, v \in C(\mathbb{R}, L_+^2) \cap C_{w*}(\mathbb{R}, \text{BMO}_+(\mathbb{T}))$. Indeed, the only technical point is to extend the identity

$$\Pi(w^2 \bar{h}) = wH_w(h) + H_w(\bar{w}h) - H_w H_1 H_w(h)$$

to the case $w, h \in \text{BMO}_+$. This can be easily achieved by approximation of w . This leads to estimate (5). Applying this estimate to $u_0 = v_0$, we conclude that there exists only one solution $u \in C(\mathbb{R}, L_+^2) \cap C_{w*}(\mathbb{R}, \text{BMO}_+(\mathbb{T}))$ of (4).

As for the conservation of the BMO norm, it is enough to observe that, given $T \in \mathbb{R}$, that we already have an inequality,

$$\|u(T)\|_{\text{BMO}} \leq \|u_0\|_{\text{BMO}}.$$

Now, precisely from what we did, the problem

$$i\partial_t v = \Pi(|v|^2 v), \quad v(0) = u(T)$$

has only one solution $v \in C(\mathbb{R}, L_+^2)$ and locally bounded in BMO, and $\|v(t)\|_{\text{BMO}} \leq \|v(0)\|_{\text{BMO}}$. Therefore $v(t) = u(t + T)$, and applying the above inequality at $t = -T$ yields $\|u_0\|_{\text{BMO}} \leq \|u(T)\|_{\text{BMO}}$, whence the desired equality.

3. PROOF OF COROLLARY 1

In the case $s \geq 1/2$, Corollary 1 is just a consequence of the uniqueness of the Cauchy problem in Theorem 1 and of the wellposedness theory in H^s [5].

In the case $0 < s < 1/2$, a first idea is to combine the stability estimate (5), the invariance of the flow by translation on \mathbb{T} , and the following representation of the H^s norm,

$$\|u\|_{H^s}^2 = \|u\|_{L^2}^2 + \int_{-1}^1 \int_{\mathbb{T}} \frac{|u(x+h) - u(x)|^2}{|h|^{1+2s}} dx dh .$$

However, this provides a result which does not take into account the conservation of the $H^{1/2}$ norm. Therefore we prefer to use the following interpolation argument, which was suggested to us by D. Tataru. Given $\lambda > 1$, one can decompose $u_0 \in H^s$ as

$$u_0 = u_0^{<\lambda} + u_0^{>\lambda} ,$$

with $\|u_0^{<\lambda}\|_{\text{BMO}} \lesssim 1$,

$$\|u_0^{<\lambda}\|_{H^{1/2}} \lesssim \lambda^{\frac{1}{2}-s} , \quad \|u_0^{>\lambda}\|_{L^2} \lesssim \lambda^{-s} .$$

Then, by the conservation of the $H^{1/2}$ norm, $u^{<\lambda} := Z(u_0^{<\lambda})$ satisfies

$$\|u^{<\lambda}(t)\|_{H^{1/2}} \lesssim \lambda^{\frac{1}{2}-s} ,$$

while the stability estimate (5) yields, with $\alpha(t) = e^{-K|t|}$ and $K = K(\|u_0\|_{\text{BMO}})$,

$$\|u(t) - u^{<\lambda}(t)\|_{L^2} \lesssim \|u_0 - u_0^{<\lambda}\|_{L^2}^{\alpha(t)} \lesssim \lambda^{-s\alpha(t)} .$$

Therefore the dyadic component $\Delta_k u(t)$ of $u(t)$ can be estimated, for every $\lambda > 0$, as

$$\|\Delta_k u(t)\|_{L^2} \lesssim 2^{-k/2} \lambda^{\frac{1}{2}-s} + \lambda^{-s\alpha(t)} .$$

Choosing $\lambda = \lambda(k, t)$ optimally, we obtain

$$\|\Delta_k u(t)\|_{L^2} \lesssim 2^{-ks\alpha(t)/(1-2s+2s\alpha(t))} ,$$

and therefore $u(t) \in H^{s(t)}$ with

$$s(t) = \frac{se^{-\tilde{K}|t|}}{1 - 2s + 2se^{-\tilde{K}|t|}}$$

for every $\tilde{K} > K$. This completes the proof.

4. PROOF OF THEOREM 2

The arguments for Theorem 2 are an adaptation of a method developed by Elgindi and Masmoudi in [2], which leads to ill-posedness for the incompressible Euler equation at the C^1 regularity. The crucial step is the following lemma.

Lemma 2. *Let $u_0 \in C_+(\mathbb{T})$. There exists a sequence (u^n) of smooth solutions to the (1) such that*

$$\|u^n(0) - u_0\|_{L^\infty} \rightarrow 0 ,$$

and a sequence of times $T_n > 0$ tending to 0 such that

$$\sup_{t \in [0, T_n]} \|u^n(t)\|_{L^\infty} \rightarrow \infty .$$

Let us show how Lemma 2 implies Theorem 2. For every $u_0 \in \text{BMO}_+(\mathbb{T})$ and every $t \in \mathbb{R}$, we denote by $Z(t)(u_0)$ the value $u(t)$ at time t of the solution $u := Z(u_0)$ of (4) provided by Theorem 1. For every integer $p \geq 1$, denote by Ω_p the subset of those $u_0 \in C_+(\mathbb{T})$ such that, for some $r_p \in]0, 1[$, we have

$$\sup_{t \in [0, 1/p]} \sup_{x \in \mathbb{T}} |Z(t)(u_0)(r_p e^{ix})| > p .$$

We claim that Ω_p is an open subset of $C_+(\mathbb{T})$. Indeed, for every $r < 1$, the map

$$u \in L_+^2(\mathbb{T}) \rightarrow u_r \in L_+^\infty(\mathbb{T}) , \quad u_r(e^{ix}) := u(re^{ix})$$

is continuous in view of the Cauchy integral formula, and the mapping

$$u_0 \in C_+(\mathbb{T}) \mapsto Z(u_0) \in C([0, 1], L_+^2(\mathbb{T}))$$

is continuous in view of Theorem 1.

Next we claim that Ω_p is dense in $C_+(\mathbb{T})$. Given $u_0 \in C_+(\mathbb{T})$, we apply Lemma 2. The sequence provided by this lemma converges to u_0 in $C_+(\mathbb{T})$. Furthermore, for n big enough, $T_n < 1/p$ and

$$\sup_{t \in [0, T_n]} \|u^n(t)\|_{L^\infty} > p .$$

Since, for every $f \in L_+^\infty(\mathbb{T})$,

$$\|f\|_{L^\infty} = \sup_{r < 1} \sup_{x \in \mathbb{T}} |f(re^{ix})| ,$$

we conclude that u^n belongs to Ω_p .

Introduce

$$\mathcal{G} = \bigcap_{p \geq 1} \Omega_p .$$

Since $C_+(\mathbb{T})$ is a Banach space, the Baire theorem shows that \mathcal{G} is a dense G_δ subset of $C_+(\mathbb{T})$. Furthermore, if $u_0 \in \mathcal{G}$, we have, for every $T > 0$ and every $p \geq T^{-1}$,

$$\sup_{t \in [0, T]} \sup_{r \in [0, 1]} \sup_{x \in \mathbb{T}} |Z(t)u_0(re^{ix})| > p ,$$

hence $Z(u_0) \notin L^\infty([0, T] \times \mathbb{T})$.

4.1. **Proof of Lemma 2.** We shall make use of a Banach algebra B of functions on the torus, invariant by Π , included into L^∞ , such that

$$(11) \quad \|uv\|_B \leq C(\|u\|_{L^\infty}\|v\|_B + \|u\|_B\|v\|_{L^\infty}) ,$$

and which, roughly speaking, has the same scaling properties as L^∞ . An example is provided by the Besov space

$$B = B_{2,1}^{1/2} = \{u \in L^2(\mathbb{T}) : \|u\|_B = |\widehat{u}(0)| + \sum_{k=0}^{\infty} 2^{k/2} \|\Delta_k u\|_{L^2} < \infty\} ,$$

where $\Delta_k u$ denotes the usual dyadic component of u . Indeed, $\Pi(B) \subset B$ from the definition, the inclusion $B \subset L^\infty$ is a consequence of the standard inequality

$$\|\Delta_k u\|_{L^\infty} \lesssim 2^{k/2} \|\Delta_k u\|_{L^2} ,$$

and the tame estimate (11) follows from parilinearising the product uv . The subspace $B_+ = \Pi(B)$ of L_+^∞ can also be characterised by the condition

$$(12) \quad [u]_{B_+} := \int_0^1 \frac{1}{\sqrt{1-r}} \left(\int_0^{2\pi} |u'(re^{ix})|^2 dx \right)^{1/2} dr < \infty ,$$

where u' is the holomorphic derivative of u , the norm $|\widehat{u}(0)| + [u]_{B_+}$ being equivalent to $\|u\|_B$ on B_+ .

We now fix $\alpha \in]0, \infty[$ and introduce, for every $\rho \in]0, 1[$,

$$f_\rho(z) = (1 - \rho z)^{i\alpha} = e^{i\alpha \log(1 - \rho z)} , |z| \leq 1 ,$$

with $\log(1 - \rho z) \in \mathbb{R} + i[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Lemma 3. *The following estimates hold as ρ tends to 1,*

$$\|f_\rho\|_{L^\infty} \leq C , \quad \|f_\rho\|_B \leq C \log \frac{1}{1 - \rho} ,$$

and for every trigonometric polynomial $g = g(z) \in L_+^2$ with $g(1) \neq 0$,

$$\|\Pi(|f_\rho|^2 g)\|_{L^\infty} \geq c(g) \log \frac{1}{1 - \rho} ,$$

for some $c(g) > 0$.

Proof. Notice that, for $x \in \mathbb{T}$,

$$f_\rho(e^{ix}) = e^{i\frac{\alpha}{2} \log(1 + \rho^2 - 2\rho \cos x)} e^{-\alpha A_\rho(x)} , \quad A_\rho(x) = \arctan \left(\frac{\rho \sin x}{1 - \rho \cos x} \right) .$$

In particular,

$$\|f_\rho\|_{L^\infty} \leq e^{\alpha\pi/2} .$$

On the other hand,

$$f'_\rho(z) = -i\alpha\rho(1 - \rho z)^{i\alpha-1} ,$$

so that

$$\int_0^{2\pi} |f'_\rho(re^{ix})|^2 dx \lesssim \frac{1}{1-\rho r},$$

and

$$[f_\rho]_{B_+} \lesssim \int_0^1 \frac{dr}{\sqrt{(1-r)(1-\rho r)}} \lesssim \log \frac{1}{1-\rho}.$$

It remains to prove the last statement. Let $g = g(z) \in L^2_+$ be a trigonometric polynomial. We compute

$$\begin{aligned} \Pi(|f_\rho|^2 g)(\rho) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|f_\rho(e^{ix})|^2 g(e^{ix})}{1-\rho e^{-ix}} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-2\alpha A_\rho(x)} g(e^{ix})}{1-\rho e^{-ix}} dx. \end{aligned}$$

The above integral is uniformly bounded as ρ tends to 1, except for the contribution of a neighborhood of $x = 0$. Symmetrizing the integration domain, we get

$$\Pi(|f_\rho|^2 g)(\rho) = \int_0^\pi \frac{h_\rho(x) + h_\rho(-x)}{(1-\rho)^2 + 2\rho(1-\cos x)} \frac{dx}{2\pi},$$

with

$$h(x) := (1-\rho e^{ix}) e^{-2\alpha A_\rho(x)} g(e^{ix}).$$

Expanding e^{ix} near $x = 0$, we obtain,

$$\Pi(|f_\rho|^2 g)(\rho) = O(1) + \frac{i\rho}{2\pi} g(1) \int_0^\pi \frac{x (e^{2\alpha A_\rho(x)} - e^{-2\alpha A_\rho(x)})}{(1-\rho)^2 + 2\rho(1-\cos x)} dx.$$

Notice that function A_ρ is nonnegative on $[0, \pi]$ and increasing from $x = 0$ to $x = \arccos \rho \sim \sqrt{2(1-\rho)}$. In particular, the integrand of the above integral is nonnegative, and we may restrict x to the domain of integration $[1-\rho, \sqrt{1-\rho}]$, on which $A_\rho(x) \gtrsim \frac{\pi}{4}$, so that

$$\begin{aligned} \int_0^\pi \frac{x (e^{2\alpha A_\rho(x)} - e^{-2\alpha A_\rho(x)})}{(1-\rho)^2 + 2\rho(1-\cos x)} dx &\geq c_\alpha \int_{1-\rho}^{\sqrt{1-\rho}} \frac{x}{(1-\rho)^2 + x^2} dx \\ &\geq \tilde{c}_\alpha \log \frac{1}{1-\rho}. \end{aligned}$$

This completes the proof of Lemma 3.

Next, we consider, for a given trigonometric polynomial $g = g(z) \in L_+^2$ such that $g(1) \neq 0$, the family of data

$$u_0^{\rho,\varepsilon} = g + \varepsilon f_\rho .$$

Applying Lemma 3, we observe that

$$\|u_0^{\rho,\varepsilon} - g\|_{L^\infty} = O(\varepsilon) , \quad \|u_0^{\rho,\varepsilon}\|_B \lesssim O(1) + \varepsilon \log \frac{1}{1-\rho} .$$

Furthermore,

$$\begin{aligned} \Pi(|u_0^{\rho,\varepsilon}|^2 u_0^{\rho,\varepsilon}) &= \Pi(|g|^2 g) + \varepsilon[2\Pi(|g|^2 f_\rho) + \Pi(g^2 \bar{f}_\rho)] + \\ &\quad + \varepsilon^2[2\Pi(|f_\rho|^2 g) + \Pi(f_\rho^2 \bar{g})] + \varepsilon^3 \Pi(|f_\rho|^2 f_\rho) . \end{aligned}$$

Notice that, if $h \in L_+^\infty$,

$$\Pi(e^{-ix} h) = e^{-ix}(h - h(0))$$

belongs to L_+^∞ with $\|\Pi(e^{-ix} h)\|_{L^\infty} \leq 2\|h\|_{L^\infty}$. Since $\Pi(|g|^2 f_\rho)$ is a finite linear combination of terms of the form $e^{inx} f_\rho$ and $\Pi(e^{-inx} f_\rho)$ with $|n|$ not greater than the degree of g , we conclude that $\Pi(|g|^2 f_\rho)$ is bounded in L_+^∞ . Similarly, $\Pi(f_\rho^2 \bar{g})$ is bounded in L_+^∞ , and so is $\Pi(g^2 \bar{f}_\rho)$, since it is a finite trigonometric polynomial of degree not greater than twice the degree of g , with coefficients estimated by the supremum of Fourier coefficients of f_ρ . Finally, applying (11) and Lemma 3,

$$\|\Pi(|f_\rho|^2 f_\rho)\|_{L^\infty} \lesssim \|\Pi(|f_\rho|^2 f_\rho)\|_B \lesssim \|f_\rho\|_B \lesssim \log \frac{1}{1-\rho} .$$

This leads to

$$\|\Pi(|u_0^{\rho,\varepsilon}|^2 u_0^{\rho,\varepsilon})\|_{L^\infty} \geq \varepsilon^2(c(g) - \varepsilon C(g)) \log \frac{1}{1-\rho} - O(1) .$$

Choosing ε small enough, we infer

$$(13) \quad \|\Pi(|u_0^{\rho,\varepsilon}|^2 u_0^{\rho,\varepsilon})\|_{L^\infty} \geq \varepsilon^2 \tilde{c}(g) \log \frac{1}{1-\rho} - O(1) , \quad \tilde{c}(g) > 0 .$$

Next we consider $u^{\rho,\varepsilon} = Z(u_0^{\rho,\varepsilon})$. We claim that, for every positive time $T \ll 1$, there exists $\rho = \rho(\varepsilon, T)$ such that, for $\varepsilon \ll 1$,

$$\limsup_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} \|u^{\rho(\varepsilon, T), \varepsilon}\|_{L^\infty} = +\infty .$$

Indeed, assume by contradiction that, for some $T > 0$ and for some M , we have, for some $\varepsilon_0 > 0$,

$$\sup_{\varepsilon < \varepsilon_0} \sup_{\rho < 1} \sup_{t \in [0, T]} \|u^{\rho, \varepsilon}\|_{L^\infty} \leq M .$$

Then, from the equation

$$u^{\rho, \varepsilon}(t) = u_0^{\rho, \varepsilon} - i \int_0^t \Pi(|u^{\rho, \varepsilon}(s)|^2 u^{\rho, \varepsilon}(s)) ds$$

and using (11), we have, if $t \in [0, T]$,

$$\sup_{s \in [0, t]} \|u^{\rho, \varepsilon}(s)\|_B \leq \|u_0^{\rho, \varepsilon}\|_B + CM^2 t \sup_{s \in [0, t]} \|u^{\rho, \varepsilon}(s)\|_B,$$

so that, if $t \leq \tilde{T}^* := \min(T, (2CM^2)^{-1})$,

$$(14) \quad \sup_{s \in [0, t]} \|u^{\rho, \varepsilon}(s)\|_B \leq 2\|u_0^{\rho, \varepsilon}\|_B \lesssim O(1) + \varepsilon \log \frac{1}{1 - \rho}.$$

Then we write the Taylor formula at second order in t ,

$$u^{\rho, \varepsilon}(t) = u_0^{\rho, \varepsilon} - it\Pi(|u_0^{\rho, \varepsilon}|^2 u_0^{\rho, \varepsilon}) + \int_0^t (t-s) [-2(T_{|u^{\rho, \varepsilon}(s)|^2})^2 + H_{u^{\rho, \varepsilon}(s)} T_{|u^{\rho, \varepsilon}(s)|^2}] u^{\rho, \varepsilon}(s) ds,$$

so that, using again (11) and (14), for every $t \in [0, T^*]$,

$$\|u^{\rho, \varepsilon}(t) - u_0^{\rho, \varepsilon} + it\Pi(|u_0^{\rho, \varepsilon}|^2 u_0^{\rho, \varepsilon})\|_B \leq K(M)\varepsilon t^2 \log \frac{1}{1 - \rho} + O(1).$$

Using (13), we infer

$$\forall t \in [0, T^*], \|u^{\rho, \varepsilon}(t)\|_{L^\infty} \geq t\varepsilon \log \frac{1}{1 - \rho} (\tilde{c}(g)\varepsilon - tK(M)) - O(1).$$

Choosing $t = T^{**} := \min(T^*, \varepsilon \tilde{c}(g)/2K(M))$ and $\rho = \rho(\varepsilon, T)$ close enough to 1, we obtain a contradiction.

Summing up, we have proved that, for every trigonometric polynomial $g = g(z) \in L_+^2$ such that $g(1) \neq 0$, there exists a sequence of data u_0^n converging to g in $C_+(\mathbb{T})$, and a sequence of positive times T_n converging to 0, such that

$$\sup_{t \in [0, T_n]} \|Z(t)u_0^n\|_{L^\infty} \rightarrow \infty.$$

Since any $u_0 \in C_+(\mathbb{T})$ can be approximated by a sequence of trigonometrical polynomials $g \in L_+^2$ with $g(1) \neq 0$, this completes the proof of Lemma 2. \square

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