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Strictly dissipative boundary value problems at trihedral corners

Laurence Halpern * Jeffrey Rauch †

Abstract

For time independent symmetric hyperbolic systems with elliptic generators, gluing strictly dissipative boundary conditions at a multihedral corner yields a well posed boundary value problem. Uniqueness of solutions with square integrable boundary traces is proved using the Laplace transform and an $H^{1/2}$ regularity theorem.

Keywords. Dissipative boundary value problems, symmetric hyperbolic systems, capacity

1 Introduction

This talk presents half of the paper [6]. The other half devoted to the analysis of Bérenger split equations at internal trihedral corners is described in the earlier seminar [5]. Since the detailed paper has already appeared, we describe the main ideas and emphasize a number of open problems. There is an extensive literature on problems with corners discussed in [6]. For trihedral and higher corners for hyperbolic problems much less is known.

2 Symmetric hyperbolic systems

$A_j(x)$ and $B(x)$ are C^∞ matrix valued and constant outside a compact set in $\mathbb{R}^d$. **For each x, $A_j(x)$ is hermitian symmetric.** The system

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is symmetric hyperbolic in the sense of Friedrichs [1]. Define symbols and operators

$$A(x,\xi) \;:=\; \sum_j A_j(x)\xi_j\,, \qquad G(x,\partial) \;:=\; A(x,\partial)+B(x)\,,$$

$$L := \partial_t + G(x,\partial), \quad Z(x) := B(x)+B(x)^* - \sum_j \partial_j A_j(x) = G+G^*\,.$$

Ω is a *nice* subset of $\mathbb{R}^d$ and for $x \in \partial\Omega$, $\nu(x)$ denotes the unit outward conormal. $A(x,\nu(x))$ is the symbol evaluated at outward conormal. **Assume that** $A(x,\partial)$ **is elliptic**. Therefore $A(x,\nu(x)$ is invertible. The analysis extends to the Maxwell's equations for which this hypothesis is violated by a *hidden ellipticity* argument in [6].

If $Lu=0$ *and* u *is* $H^1([0,T]\times\Omega)$ *then*

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} \;+\; \big(Z(x)u\,,\,u\big)_{L^2(\Omega)} \;=\; -\int_{\partial\Omega}\big(A(x,\nu(x))u,u\big)\,d\Sigma\,.$$

3 Dissipative conditions on smooth domains

• Homogeneous linear boundary conditions are prescribed in the form, $u(t,x)\in\mathcal{N}(x)$ for $x\in\partial\Omega$ where $\mathcal{N}(x)\subset\mathbb{C}^N$ is a smoothly varying linear subspace.

• $\mathcal{N}$ is **dissipative** when $u\in\mathcal{N}$ implies $(A(x,\nu(x))u,u)\ge 0$. It is **strictly dissipative** when $\ge c\|u\|^2$ with $c>0$.

• It is **maximal** when the dimension of $\mathcal{N}$ is equal to the the dimension of the positive spectral subspace of $A(x,\nu(x))$.

Friedrichs' Theorem [2]. *For* $g\in L^2(\Omega)$ *and a maximal dissipative boundary condition,* $\exists!$ *solution* $u\in C([0,\infty[\,:\,L^2(\Omega))$ *with initial value* g. *The solution satisfies the energy identity.*

Remark. The result is also true for time dependent $G,\mathcal{N}$.

Example. If $Z\ge 0$ then $\|u(t)\|_{L^2(\Omega)}\searrow$.

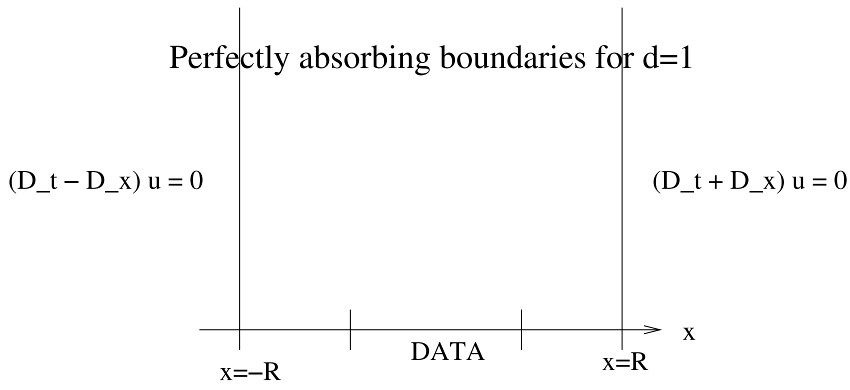
More modern treatments follow the lead of [7].

4 Corner motivation, the simplest example

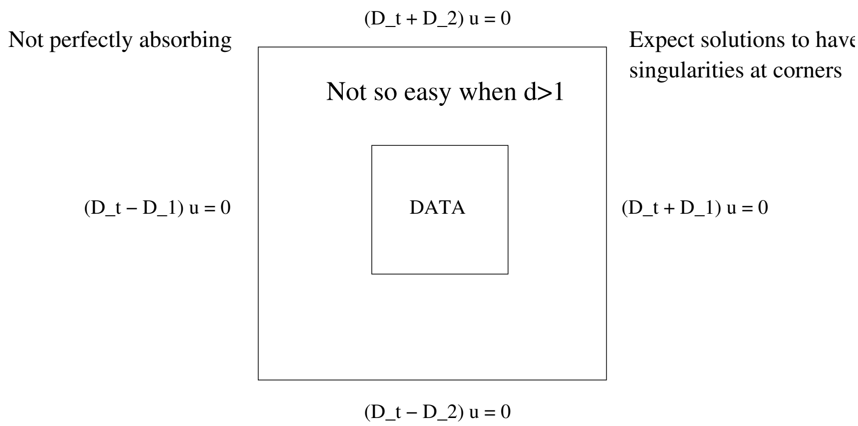
To solve numerically $\Box_{1+d}u=0$ on all of $\mathbb{R}^d$ with initial data supported in $\{|x_j|<1\}$ one often introduces a finite rectangular computational domain.

If one is interested in the solutions on $\{|x_j| < r\}$, compute on the larger rectangle $\{|x_j| < R\}$.

In case $d = 1$ this is easy to do. Thinking in terms of rightward and leftward going waves shows that the following boundary conditions yield an exact evaluation.



In higher dimensions the analogous problem is a reasonalble absorbing boundary condition but is not exact. The two dimensional problem has different boundary conditions on adjacent edges that touch.



In all dimensions, these problems are associated to a coercive elliptic quadratic form. Analysis of H^1 solutions using this structure follows the standard procedure.

No boundary condition is required at singular subset of boundary. The singular set is codimension 2 in $\mathbb{R}^d$ hence negligible for H^1.

Open problem 1. Replace the first order operators $\partial_t + \nu \cdot \partial_x$ by high order absorbing boundary condition on each face. Trihedral analysis is *terra incognita.* Numerically, they work.

Corner problems for symmetric hyperbolic systems are usually not associated with a coercive second order elliptic boundary value problem. There is no such easy analytic treatment.

The natural first case to consider is different dissipative boundary conditions at the faces of a cubic computational domain. That is the subject of this talk.

5 Geometry of a single high dimensional corner

With notation motivated by the *octant* when $d = 3$,

$$\mathcal{O} \;:=\; \Big\{ x \in \mathbb{R}^d \;:\; 0 < x_j\,, \quad \text{for}, \quad j = 1, 2, \ldots, d \Big\}\,.$$

The smooth points of the boundary are those points where *exactly one* coordinate x_j vanishes.

The singular part of the boundary, denoted $\mathcal{S}$, consist of points where *at least two* coordinates vanish.

The set of points where exactly k coordinates vanish are edges of dimension $d - k$ in $\mathcal{S}$.

6 Instructive example/counterexample

Consider the solution of the radiation problem

$$\Box_{1+d} w \;=\; f(t)\,\delta(x)\,, \qquad w = f = 0 \quad \text{for} \quad t \leq 0, \qquad f \in C^\infty\,.$$

Since δ is even in each x_j, the same is true of $f(t)\delta(x)$ and therefore w.

Evenness implies that on each face of $\mathbb{R}_t \times (\partial\mathcal{O} \setminus \mathcal{S})$, w satisfies the homogeneous Neumann boundary condition $\nu \cdot \partial_x w = 0$.

It satisfies $\Box w = 0$ in $\mathbb{R}_t \times \mathcal{O}$ and has vanishing Cauchy data for $t < 0$ and homogeneous boundary conditions, yet there are waves in $\mathcal{O}$.

The waves come out of the corner. The Neumann corner problem is well set for solutions in H^1. The example has $w \notin H^1$. In dimension $d = 2$

the solution is $H^{1-\epsilon}$ for any $\epsilon > 0$, so barely misses. To prove this note that the right hand side lies in $C^\infty(\mathbb{R}_t : H^{-d/2-\epsilon}(\mathbb{R}^d))$. The right hand side has wavefront set in $\{\tau = 0\}$ on which $\Box$ is elliptic. The solution $w \in C^\infty(\mathbb{R}_t : H^{-d/2-\epsilon}(\mathbb{R}^d)$. For $d = 2$ this yields the announced regularity.

No boundary condition needed at the points of the corner. However, H^1 regularity there is crucial for the well posedness of the Neumann problem. Without it there is nonuniqueness.

7 Main theorem

The next theorem asserts existence and uniqueness of solutions that have square integrable traces on the boundary.

Main Theorem. Assume. a. Time independent A_j and $\mathcal{N}_j$ with $\sum_j A_j(x)\partial_j$ elliptic for $x \in \partial\mathcal{O}$.

b. Subspaces $\mathcal{N}_j(x)$, defined for $x \in \{x_j = 0\}$, are smooth and maximal strictly dissipative.

Then, $\forall\, g \in L^2(\mathcal{O})$, $\exists!\, u$ satisfying three conditions.

i. For some C,

$$e^{-Ct}u \in L^2\big(]0,\infty[\times\mathcal{O})\big), \quad e^{-Ct}\,A(x,\nu(x))u|_{\partial\mathcal{O}} \in L^2(]0,\infty[\times\partial\mathcal{O})\,.$$

ii. $Lu = 0$ in $]0,\infty[\times\mathcal{O}$ and $u|_{t=0} = g$ on Ω.

iii. For each $1 \le j \le d$,

$$u|_{\{x_j=0\}\cap\{\partial\mathcal{O}\setminus\mathcal{S}\}} \;\in\; \mathcal{N}_j\,, \qquad \text{(no BC at edges/corners)}$$

In addition, for all $0 \le t < T < \infty$ the energy identity is satisfied,

$$\|u(T)\|^2 + \int\limits_{[t,T]\times\partial\mathcal{O}} \big(A\big(x,\nu(x)\big)u,u\big)\,dt\,d\Sigma + \int\limits_{[t,T]\times\mathcal{O}} \big(Z(x)u,u\big)\,dt\,dx = \|u(t)\|^2\,.$$

8 Proof of existence, the easy part

Proof. Consider a sequence of smoothed domains $\mathcal{O}^\epsilon$ obtained by rounding the edges in an ϵ neighborhood of the singular set $\mathcal{S}$.

On $\partial\mathcal{O}^\epsilon$, construct strictly dissipative boundary space $\mathcal{N}^\epsilon$ in two steps.

1. Modify the spaces $\mathcal{N}_j$ on a 4ϵ neighborhood of $\mathcal{S}$ so that they are equal to the positive spectral subspace of $A(x, \nu(x))$ on an 2ϵ neighborhood of $\mathcal{S}$.

2. Choose a boundary space on $\partial\mathcal{O}^\epsilon$ that is equal to the positive spectral subspace of $A(x, \nu(x))$ on the rounded parts of $\partial\mathcal{O}^\epsilon$ and equal to the original $\mathcal{N}_j$ outside a 4ϵ neighborhood of S.

The rounded domain has no corner so Friedrichs' Theorem constructs a solution u^ϵ with uniform L^2 estimates in $\mathcal{O}$ and uniform L^2 estimates for the trace at the boundary.

Passing to the limit $\epsilon \to 0$ constructs a solution with at most exponential growth. Satisfies energy *inequality* on $0 < t < T$.

The energy equality is recovered after uniqueness, see [6].

9 Uniqueness, the hard part

The difference of two solutions could be a solution of the homogeneous problem with waves entering from the corner.

Need to show that such waves cannot be L^2 with L^2 traces.

Open problems.

2. Do not know if uniqueness is true without the L^2 trace assumption.

3. Do not know if uniqueness is true if the ellipticity is dropped.

4. Do not know if uniqueness is true if the coefficients A_j depend also on time.

Uniqueness can *fail if one drops ellipticity* **and** *the L^2 trace*, see [8].

9.1 The $H^{1/2}$ strategy

Uniqueness of H^1 solutions is easy. Simply take the scalar product of the equation with u and integrate by parts. For $u \in H^1$ then the terms in (Lu, u) are the product of an L^2 times an H^1. This is *more* than enough to justify the integration by parts.

For $u \in H^{1/2}$, the terms are a product of an $H^{1/2}$ times an $H^{-1/2}$. This is a borderline case and it works.

Prove that solutions with zero initial data vanish **in three steps**.

1. Show that Laplace Transform of a solution with $u(0) = 0$ vanishes by showing that the energy identity holds for them.

2. Use a partition of unity. At interior points use Friedrichs' classical mollifiers. At the boundary use an elliptic argument to show that the solution is $H^{1/2}$. The L^2 trace is used for this.

3. A **capacity argument** yields the energy identity for $H^{1/2}$ solutions.

9.2 Laplace transform

Laplace transform $\widetilde{u}(\tau, x)$ satisfies

$$\tau\widetilde{u} + G(x, \partial_x)\widetilde{u} \;=\; 0\,, \qquad \widetilde{u}|_{\partial\mathcal{O}\setminus\mathcal{S}} \in \mathcal{N}\,.$$

The strategy is to take the the real part of the $L^2(\mathcal{O})$ scalar product with u. An integration by parts yields

$$\big((\mathrm{Re}\,\tau + Z)\widetilde{u}\,,\,\widetilde{u}\big) + \int_{\partial\mathcal{O}} (A(x, \nu(x))\widetilde{u}\,,\;\widetilde{u})\,d\Sigma \;=\; 0\,.$$

If this is justified it yields $\widetilde{u} = 0$ for $\mathrm{Re}\,\tau > \|Z\|_{L^\infty}$.

Decompose $\widetilde{u}$ using a partition of unity.

Friedrichs [1] treats interior points.

At points of $\partial\mathcal{O} \setminus \mathcal{S}$, $\widetilde{u} \in C^\infty$ since *strictly dissipative boundary conditions satisfy elliptic lopatinski for the transformed equation.*

The hard part is to justify integration by parts for pieces of $\widetilde{u}$ touching $\mathcal{S}$.

9.3 Extension by zero and parametrix

Justify integration by parts for $\tau + G$ and v supported compactly near $\mathcal{S}$ satisfying

$$v \in L^2(\mathcal{O}), \quad Gv \in L^2(\mathcal{O}), \quad A(x, \nu(x))\, v\big|_{\partial\mathcal{O}} \in L^2(\partial\mathcal{O} \setminus \mathcal{S})\,.$$

Key step is to show that $v \in H^{1/2}(\mathcal{O})$.

Denote by $\underline{v}$ the extension by zero. Then

$$G\underline{v} \;=\; \underline{f} \;+\; A(x, \nu(x))v|_{\partial\mathcal{O}\setminus\mathcal{S}}\; d\Sigma\,, \qquad \underline{f} \in L^2(\mathcal{O})$$

Choose a classical properly supported pseudodifferential parametrix $P(x, D) \in \mathrm{Op}(S^{-1}(\mathbb{R}^d))$ of G on a neighborhood of $\mathcal{S}$. Then

$$P(x, D)(f + A(x, \nu(x))v|_{\partial\mathcal{O}} d\Sigma) \;-\; \underline{v} \;\in\; \cap_s H^s(\mathbb{R}^d)\,.$$
$$P \in \mathrm{Op}(S^{-1}) \quad\Longrightarrow\quad Pf \;\in\; H^1(\mathbb{R}^d)\,.$$

9.4 Regularity modulo a layer potential

Denote $H_j := \{x_j > 0\}$. Decompose

$$A(x,\nu(x))\,v|_{\partial\mathcal{O}\setminus\mathcal{S}} \;=\; \sum g_j\,, \qquad g_j \;\in\; L^2_{compact}(\partial H_j \cap \partial\mathcal{O})\,.$$

To show $v \in H^{1/2}(\mathcal{O})$, suffices to show the layer potentials $w_j := P(x,D)(g_j\,d\Sigma)$ belong to $H^{1/2}(\mathcal{O})$. Prove the stronger assertions, $w_j \in H^{1/2}(H_j)$.
For $p := 2d/(d{+}1) < 2$, the trace of $W^{1,p}(\mathbb{R}^d)$ belongs to $L^2(\partial H_j)$. Therefore

$$g_j\,d\Sigma \in \big(W^{1,p}(\mathbb{R}^d)\big)' = W^{-1,q}(\mathbb{R}^d), \qquad \frac{1}{q} + \frac{1}{p} = 1, \quad q > 2\,.$$

9.5 Regularity of the layer potential

$P \in \mathrm{Op}S^{-1}$ properly supported and $g_j\,d\Sigma \in W^{-1,q}(\mathbb{R}^d)$ yield

$$w_j := P(x,D)\big(g_j\,d\Sigma\big) \in W^{0,q}_{cpct}(\mathbb{R}^d) = L^q_{cpct}(\mathbb{R}^d) \subset L^2_{cpct}(\mathbb{R}^d)$$

In addition

$$Gw_j \;-\; g_j\,d\Sigma \;:=\; h_j \;\in\; \cap_s H^s(\mathbb{R}^d)$$

Since $g_j\,d\Sigma$ vanishes on $\mathcal{O}$

$$G(w_j)|_{H_j} \;=\; h_j|_{H_j} \;\in\; \cap_s H^s(H_j) \qquad (*)$$

Principal symbol of G is odd in $\xi_j \Rightarrow$ principal symbol $P_{-1} = G_1^{-1}$ is odd. This is the transmission condition for ∂H_j.
The transmission condition and $g_j \in L^2(\partial H_j)$ yield

$$w_j|_{\partial H_j} \;=\; P(x,D)\big(g_j\,d\Sigma\big)|_{\partial H_j} \;\in\; L^2(\partial H_j) \qquad (**)$$

The overdetermined boundary value problem $(*), (**)$ is coercive. Elliptic regularity implies $w_j|_{H_j} \in H^{1/2}(H_j)$.

Open problem 5. Find regularity theorems à la Grisvard [3],[4] asserting that modulo finite dimensional sets of singular corrector functions there is standard elliptic gain of one.

10 Final capacity argument

Choose ψ^ϵ cutoff functions vanishing on an $\epsilon/2$ neighborhood of $\mathcal{S}$ and equal to one outside an ϵ neighborhood. The energy identity for $\psi^\epsilon v$ is satisfied as there is no corner

$$\mathrm{Re}\big(G\psi^\epsilon v, \psi^\epsilon v\big) + \big(Z\psi^\epsilon v, \psi^\epsilon v\big) = -\int_{\partial\mathcal{O}} \big(A(x,\nu(x))\psi^\epsilon v, \psi^\epsilon v\big) d\Sigma\,.$$

Passing to the limit, the troublesome terms occur when the derivatives hit ψ^ϵ. They are

$$\lesssim \int_{\mathrm{dist}\,(x,\mathcal{S})\le\epsilon} \frac{1}{\epsilon}\ |v|^2\ dx\,.$$

This tends to zero as $\epsilon \to 0$ *because* $v \in H^{1/2}$ *on a neighbhorhood of* $\mathcal{S}$ *in* $\mathcal{O}$.

Passing to the limit justifies the energy identity for the parts of $\widetilde{u}$ near $\mathcal{S}$. Summing all the local energy identities yields the global identity. That implies $\widetilde{u}(\tau) = 0$ for $\mathrm{Re}\,\tau > \|Z\|_{L^\infty}$. □

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