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A NONLINEAR FOURIER TRANSFORM FOR THE BENJAMIN–ONO EQUATION ON THE TORUS AND APPLICATIONS

PATRICK GÉRARD

Abstract. The Benjamin-Ono equation was introduced by Benjamin in 1967 as a model for a special regime of internal gravity waves at the interface of two fluids. This nonlinear dispersive equation admits a Lax pair structure involving nonlocal operators of Toeplitz type on the Hardy space. In the case of periodic boundary conditions, the spectral study of these Lax operators allows us to construct a nonlinear Fourier transform which conjugates the Benjamin–Ono dynamics to advection with constant velocity on tori. This construction has several applications: low regularity well-posedness of the initial value problem, long time behaviour of solutions and stability of traveling waves. This is a short report on these results, recently obtained in collaboration with T. Kappeler and P. Topalov.

1. Introduction

The Benjamin–Ono equation [5], [23], reads

$$\partial_t u = H\partial_x^2 u - \partial_x(u^2), \tag{1}$$

where $u = u(t,x)$ is a real valued function and H denotes the Hilbert transform. We refer to the recent survey by Saut [25] and to references therein for a discussion of the origin of this equation as a model for long, one-way internal gravity waves in a two-layer fluid, and for a comprehensive bibliography. The two natural possible assumptions on the unknown u as a function of the real variable x are either some decay at infinity, for instance square integrability on the whole line, or periodicity. Our results apply to the latter, so we shall assume $x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, therefore H takes the form

$$\widehat{Hf}(n) = -i\,\mathrm{sign}(n)\widehat{f}(n)$$

where $\widehat{f}(n)$, $n \in \mathbb{Z}$, denote the usual Fourier coefficients of the 2π–periodic function f, and $\mathrm{sign}(0) := 0$.
For Equation (1), the following issues naturally arise as relevant problems.

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- Wellposedness theory: on which Sobolev spaces $H^s(\mathbb{T},\mathbb{R})$ does (1) define a continuous flow map?
- Traveling waves: what are the traveling waves solutions of (1)? Are they orbitally stable?
- Long time behaviour: what are the global properties of the trajectories of (1)? In which Sobolev spaces are they bounded? Relatively compact? Do they satisfy a Poincaré recurrence theorem?

These issues have been addressed for decades by using various methods. Here our purpose is to show that an appropriate use of the integrability of (1) leads to optimal answers to these three problems.

1.1. **Sharp wellposedness theory.** The wellposedness theory of the initial value problem for (1) has been extensively studied for forty years. The first results are due to Saut [24] and Abdelouhab, Bona, Felland and Saut [1] and state that, for s large enough, for every $u_0 \in H^s(\mathbb{T},\mathbb{R})$, there exists a unique solution $u \in C(\mathbb{R}, H^s(\mathbb{T},\mathbb{R}))$ of (1) satisfying $u(0,x) = u_0(x)$. This result is based on the combination of energy estimates and appropriate conservation laws and leads to the definition of a continuous flow map $S(t) : H^s(\mathbb{T},\mathbb{R}) \to H^s(\mathbb{T},\mathbb{R})$ by $S(t)u_0 = u(t,.)$. Then, by means of dispersive estimates and of a gauge transformation introduced by Tao [28], several improvements were successively obtained, establishing the continuous extension of $S(t)$ to H^s for smaller and smaller values of s. The best results in this direction are due to Molinet [19] and Molinet and Pilod [20] and state that such a continuous extension holds for every $s \geq 0$. On the other hand, the equation on the line enjoys a scaling symmetry which preserves the homogeneous H^s norm for $s = -1/2$. This suggests that such a continuous extension of $S(t)$ does not hold for $s < -1/2$. This fact was proved by Angulo Pava and Hakkaev in [4], using traveling wave solutions. The following result fills the gap between $s = -1/2$ and $s = 0$.

Theorem 1. [14] *For every $s > -1/2$, for every $t \in \mathbb{R}$, $S(t)$ extends as a continuous map from $H^s(\mathbb{T},\mathbb{R})$ to $H^s(\mathbb{T},\mathbb{R})$, so that the mapping*

$$(t,u) \in \mathbb{R} \times H^s(\mathbb{T},\mathbb{R}) \longmapsto S(t)u \in H^s(\mathbb{T},\mathbb{R})$$

is continuous. Furthermore, this property fails for $s = -1/2$ in the following sense: there exists a sequence $u^{(k)}$ of smooth initial data, converging to 0 in $H^{-1/2}(\mathbb{T},\mathbb{R})$, such that the sequence of functions $t \mapsto \langle S(t)u^{(k)} | \, \mathrm{e}^{ix} \rangle$ does not converge pointwise to 0 on any given time interval of positive length.

Notice that, if u is just H^s with $s < 0$, u^2 may not be defined as a distribution, and the formulation (1) may not have a meaning. However, the above theorem states that, if $s > -1/2$, the evolution $S(t)u$ can be defined unambiguously as the limit in H^s of $S(t)u^{(k)}$, for any sequence $u^{(k)}$ of sufficiently smooth data converging to u in H^s.

1.2. **Traveling waves and orbital stability.** A traveling wave of (1) is a solution of the form

$$u(t,x) = U(x - ct)$$

for some real number c called the velocity of the traveling wave. The traveling wave profile U satisfies the elliptic equation

$$HU'' - (U^2)' + cU' = 0 \tag{2}$$

so that, if $U \in L^2(\mathbb{T})$, U is smooth. Benjamin observed that examples of such profiles are Poisson kernels,

$$U_r(x) = \frac{1-r^2}{1-2r\cos(x)+r^2},\ c_r = \frac{1+r^2}{1-r^2},$$

where $r \in (0,1)$ is arbitrary.
Using a formulation of Equation (2) as a nonlinear ODE in the complex domain, Amick and Toland [2] classified the traveling wave profiles as

$$U(x) = N\,U_r(Nx + \alpha) + a \tag{3}$$

for some positive integer N and some constant $a \in \mathbb{R}$. Later, Angulo Pava and Natali [3] proved that every such traveling wave is orbitally stable in $H^{1/2}$. We claim that these properties extend to H^s for any $s > -1/2$. Again, notice that the profile equation (2) does not have a meaning in negative regularity, so we prefer to define a traveling wave profile as a distribution

$$U \in \bigcup_{s>-1/2} H^s(\mathbb{T},\mathbb{R})$$

such that, for some $c \in \mathbb{R}$,

$$\forall t \in \mathbb{R},\ S(t)U = U(. - ct),$$

where $S(t)$ is defined by Theorem 1.

Theorem 2. [14] *The traveling wave profiles are given by* (3) *and are orbitally stable in H^s for every $s > -1/2$. Namely, if U is given by* (3)*, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $u_0 \in H^s(\mathbb{T},\mathbb{R})$,*

$$\|u_0 - U\|_{H^s} \le \delta \Longrightarrow \sup_{t\in\mathbb{R}} \inf_{\alpha\in\mathbb{T}} \|S(t)u_0 - U(. + \alpha)\|_{H^s} \le \varepsilon.$$

As usual in stability theory, the infimum on translations by $\alpha \in \mathbb{T}$ is necessary to ensure stability, since two trajectories with close initial data cannot stay uniformly close, as shown by the example of two traveling waves $U_r(x - c_r t)$ associated to two close values of r.

1.3. **Long time behaviour.** We now come to the more general problem of describing long time behaviour of trajectories of the dynamical system defined by the flow map $S(t)$ associated to (1) thanks to Theorem 1. Let $u_0 \in H^s(\mathbb{T}, \mathbb{R})$.
The first natural question about the trajectory $(S(t)u_0)_{t\in\mathbb{R}}$ is its boundedness in H^s. If $s = 0$, this trivially holds since the L^2 norm is a conservation law for (1). More generally, it is known since Bock and Kruskal [7] and Nakamura [22] that (1) enjoys a sequence of conservation laws $(\mathcal{E}_k)_{k\in\mathbb{Z}_+}$ such that, for every $k \in \mathbb{Z}_+$, the boundedness of $\mathcal{E}_j(u)$ for $j = 0, \dots, k$ is equivalent to the boundedness of $\|u\|_{H^{k/2}}$. This implies boundedness of trajectories in H^s when s is half an integer. What about the other values of $s \in (-1/2, +\infty)$? In the special case $s \in (1/2, 1)$, an attempt was made in [16] where polynomial upper bounds were derived for the H^s norm of $S(t)u_0$, by using some nonlinear smoothing property. In fact, we are going to prove that, for any $s \in (-1/2, +\infty)$, the trajectory $S(t)u_0$ is bounded in H^s. Let us mention that H^s bounds for smooth solutions of (1) have been independently derived by Talbut [27], using the method of perturbation determinants of [17]. Combining this bound with Theorem 1 would also lead to the boundedness of all trajectories in H^s.
Another natural issue is the Poincaré recurrence property: given $u_0 \in H^s(\mathbb{T}, \mathbb{R})$, does there exist a sequence $t_n \to +\infty$ such that $S(t_n)u_0 \to u_0$ in H^s? In a series of papers, [29], [30], [31], [32], [9], Tzvetkov, Visciglia and Deng constructed invariant measures by the flow of (1) with decreasing regularity. As a consequence, they prove the recurrence property for almost every data. Notice that the lowest regularity was obtained in [9] and corresponds to the intersection of Sobolev spaces with negative regularity. We shall extend this recurrence property to all trajectories in H^s for every $s > -1/2$ by answering a question posed in [31].

Theorem 3. [14] *For every $s > -1/2$, for every $u_0 \in H^s(\mathbb{T}, \mathbb{R})$, the function*

$$t \in \mathbb{R} \mapsto S(t)u_0 \in H^s(\mathbb{T})$$

is almost periodic.

Recall that a continuous function $f : \mathbb{R} \to E$, where E is a Banach space, is almost periodic if for every $\varepsilon > 0$, there exists $\ell > 0$ such that every interval of length ℓ contains some real τ such that

$$\forall t \in \mathbb{R},\ \|f(t + \tau) - f(t)\|_E \leq \varepsilon.$$

According to Bochner's criterion [18], almost periodic functions are characterized as those continuous functions $f : \mathbb{R} \to E$ such that the set of translations $\{f(. + h), h \in \mathbb{R}\}$ is relatively compact for the norm of uniform convergence on the space of bounded continuous functions from $\mathbb{R}$ to E.

Note that Theorem 3 not only implies the boundedness of H^s trajectories, but their relative compactness in H^s.

1.4. **The nonlinear Fourier transform.** In fact, Theorems 1, 2 and 3 are easy consequences of the existence of a non linear Fourier transform for (1). Before stating this result, let us introduce some more notation. Firstly, we observe that, if u is a solution of (1) and $a \in \mathbb{R}$, then

$$u_a(t,x) := a + u(t, x + 2at) \tag{4}$$

is a solution of (1). Since, moreover, the average of u is conserved along the trajectories of (1), we may reduce the study to solutions of average 0. Given $s \in \mathbb{R}$, we define

$$H^s_{r,0} := \left\{ u \in H^s(\mathbb{T}, \mathbb{R}) : \int_{\mathbb{T}} u(x)\, dx = 0 \right\}.$$

On the other hand, given $\sigma \in \mathbb{R}$, we define

$$h^\sigma_+ := \left\{ (\zeta_n)_{n\geq 1} : \zeta_n \in \mathbb{C},\ \sum_{n=1}^{\infty} n^{2\sigma} |\zeta_n|^2 < +\infty \right\}.$$

Our main result can now be stated.

Theorem 4. [14] *There exists a mapping*

$$\begin{aligned} \Phi : \bigcup_{s>-1/2} H^s_{r,0} \quad &\longrightarrow \quad \bigcup_{s>-1/2} h^{1/2+s}_+ \\ u \quad &\longmapsto \quad (\zeta_n(u))_{n\geq 1} \end{aligned}$$

with the following properties.
(i) For every $s > -1/2$, $\Phi : H^s_{r,0} \to h^{s+1/2}_+$ is a homeomorphism exchanging bounded subsets.
(ii) For every $u_0 \in H^s_{r,0}$ with s large enough,

$$\forall n \geq 1,\ \zeta_n(S(t)u_0) = \mathrm{e}^{it\omega_n(u_0)} \zeta_n(u_0),$$
$$\omega_n(u_0) := n^2 - 2 \sum_{k=1}^{\infty} \min(k,n) |\zeta_k(u_0)|^2.$$

(iii) For every $\tau \in \mathbb{T}$, for every $u \in \bigcup_{s>-1/2} H^s_{r,0}$,

$$\forall n \geq 1,\ \zeta_n(u(. + \tau)) = \mathrm{e}^{in\tau} \zeta_n(u).$$

Note that every $u \in H^s_{r,0}$ is characterized by the sequence

$$\xi_n(u) := n^{-1/2} \widehat{u}(n),\ n \geq 1$$

which belongs to $h^{s+1/2}_+$. Furthermore, the map $u \mapsto (\xi_n(u))_{n\geq 1}$ is an isomorphism between $H^s_{r,0}$ and $h^{s+1/2}_+$ satisfying property (iii) above and property (ii) when replacing $S(t)$ by the linear evolution

$$\mathrm{e}^{tH\partial_x^2}$$

obtained by neglecting the nonlinear term in (1), and replacing ω_n by n^2. The map Φ of Theorem 4 can therefore be considered as a nonlinear version of the Fourier transform, adapted to the flow $S(t)$. In fact, its similarity with the Fourier transform also appears in the Parseval formula for every $u \in L^2(\mathbb{T}, \mathbb{R})$ with average 0,

$$\frac{1}{2\pi}\int_{\mathbb{T}} |u(x)|^2\, dx = 2\sum_{n=1}^{\infty} n|\xi_n(u)|^2,$$

which has the following nonlinear version

$$\frac{1}{2\pi}\int_{\mathbb{T}} |u(x)|^2\, dx = 2\sum_{n=1}^{\infty} n|\zeta_n(u)|^2. \tag{5}$$

The latter identity is a special case of a family of identities which express the conservation laws $\mathcal{E}_k(u)$ in terms of the $|\zeta_n(u)|^2$.
As we will see in the next section, the Benjamin–Ono equation (1) is Hamiltonian with the energy $\mathcal{E}_1$ for some symplectic structure on $H^0_{r,0}$, which is sent through Φ to the Kähler structure on $h^{1/2}_+$ induced by the h^0_+ inner product. This explains the choice of the unusual normalisation $\Phi : H^s_{r,0} \to h^{s+1/2}_+$. Then quantities $|\zeta_n|^2$ can be seen as actions, and Φ conjugates the dynamics of (1) to advection with constant velocity on tori given by the equations $|\zeta_n|^2 = \gamma_n, n \geq 1$. For this reason, Φ is called a Birkhoff map in [13], [14]. Theorem 4 expresses integrability of the Benjamin–Ono equation in the strongest sense. We refer to the introduction of [14] for a comparison with other integrable partial differential equations.
In the next section, we review the main steps of the proof of Theorem 4. Then we explain how this theorem implies Theorems 1, 2 and 3. Finally we mention some topics and results in the continuation of our results.

2. Lax operators and construction of the nonlinear Fourier transform

2.1. **The Lax pair structure.** A crucial property of the Benjamin–Ono dynamics is the existence of a Lax pair, discovered by Nakamura [22], see also Fokas and Ablowitz [10]. In order to describe this structure, we introduce some notation. We denote by $L^2_+(\mathbb{T})$ the closed subspace of $L^2(\mathbb{T})$ consisting of square integrable functions f satisfying

$$\forall n < 0,\ \widehat{f}(n) = 0,$$

and by Π the orthogonal projector from $L^2(\mathbb{T})$ to $L^2_+(\mathbb{T})$, which extends as a linear map on $\mathcal{D}'(\mathbb{T})$,

$$\widehat{\Pi f}(n) = \mathbf{1}_{n\geq 0}\,\widehat{f}(n),\ n\in\mathbb{Z}.$$

If $b\in L^\infty(\mathbb{T})$, we denote by T_b the Toeplitz operator of symbol b defined on $L^2_+(\mathbb{T})$ by

$$T_b f = \Pi(bf),\ f\in L^2_+(\mathbb{T}).$$

Note that T_b is a bounded operator, and that its adjoint is $T_b^* = T_{\overline{b}}$. Let u be a solution of (1) of regularity H^s with s large enough. For every time $t\in\mathbb{R}$, the operator

$$L_{u(t)} = D - T_{u(t)}$$

where $D := -i\partial_x$, is an unbounded selfadjoint operator on L^2_+, with domain $H^1_+ := H^1\cap L^2_+$. The Lax pair identity is then[1]

$$\frac{d}{dt}L_{u(t)} = [B(t), L_{u(t)}],\ B(t) := i(T_{|D|u(t)} - T^2_{u(t)}),\tag{6}$$

where $|D|$ denotes the Fourier multiplier of symbol $|n|$. Since $B(t)$ is a continuous family of bounded antiselfadjoint operators, the initial value problem

$$\frac{dU}{dt}(t) = B(t)U(t),\ U(0) = \mathrm{Id}$$

leads to a continuous family of unitary operators $U(t)$ on L^2_+, and (6) implies

$$\forall t\in\mathbb{R},\ L_{u(t)} = U(t)L_{u(0)}U(t)^*.\tag{7}$$

Consequently, the spectrum of $L_{u(t)}$ is conserved along the Benjamin–Ono trajectories. Since $L_{u(t)}$ has compact resolvent and is bounded from below, this spectrum is made of a sequence of eigenvalues tending to $+\infty$, and each eigenvalue provides a conservation law for (1).

2.2. **The Lax operator.** Let $u\in H^s_{r,0}$ with $s>-1/2$. Let us construct a selfadjoint realisation of

$$L_u = D - T_u\tag{8}$$

on L^2_+. If $s=0$, namely $u\in L^2(\mathbb{T},\mathbb{R})$, then

$$\|\Pi(uf)\|_{L^2}\leq \|u\|_{L^2}\|f\|_{L^\infty}\leq \varepsilon\|Df\|_{L^2} + C(\varepsilon,\|u\|_{L^2})\|f\|_{L^2}$$

for every $\varepsilon>0$, and the selfadjointness of L_u with domain H^1_+ is a direct consequence of the Kato–Rellich criterion. If $s<0$, the argument has to be modified as follows. If $f,g\in H^\sigma$, with $1/4<\sigma<1/2$,

[1]The operator $B(t)$ was found in [13] and differs from the one in the original papers by a polynomial of $L_{u(t)}$. It presents the advantages of being bounded if u is smooth enough.

it is classical — see e.g. [6]—that $f\overline{g} \in H^{2\sigma-1/2}$. Consequently, if $-1/2 < s < 0$, we have, for $f, g \in H^\sigma_+ := H^\sigma \cap \mathrm{Ran}\Pi$,

$$|\langle \Pi(uf)|g\rangle| = |\langle u|\overline{f}g\rangle| \le C\|u\|_{H^s}\|f\|_{H^\sigma}\|g\|_{H^\sigma} \tag{9}$$

with

$$\sigma := \frac{|s|}{2} + \frac{1}{4} < \frac{1}{2}.$$

Therefore the Hermitian form on $H^{1/2}_+$

$$Q(f,g) = \langle Df|g\rangle - \langle \Pi(uf)|g\rangle = \langle Df|g\rangle - \langle u|\overline{f}g\rangle \tag{10}$$

is a perturbation of $\langle Df|g\rangle$ and, by the general theory of selfadjoint operators associated to Hermitian forms, one constructs a semibounded selfadjoint operator L_u by defining its domain as the space of $f \in H^{1/2}_+$ such that, for some $C > 0$,

$$\forall g \in H^{1/2}_+,\ |Q(f,g)| \le C\|g\|_{L^2},$$

and by setting $\langle L_u f|g\rangle := Q(f,g)$ (see [14] for more detail). Furthermore, since $1 + s > 1/2$, L_u extends as a continuous mapping from H^{1+s}_+ to H^s_+. In particular,

$$L_u(1) = -\Pi u.$$

2.3. **The spectral study of the Lax operator.** Since its domain is included in $H^{1/2}_+$ — in fact in H^{1+s}_+ —, the operator L_u defined by (8) has a compact resolvent, and its spectrum consists of a sequence of eigenvalues which tends to $+\infty$.
Our next step is the following gap property for these eigenvalues.

Lemma 1 ([13])**.** *The eigenvalues of L_u are simple and the gap between two of them is at least* 1*.*

Proof. The key ingredient is an identity involving the form Q defined by (10) and the shift operator on L^2_+ defined by

$$Sf(x) := \mathrm{e}^{ix}f(x).$$

Using the Leibniz formula, one easily checks that

$$Q(Sf, Sg) = Q(f,g) + \langle f|g\rangle. \tag{11}$$

Denote by $\lambda_0(u) \le \lambda_1(u) \le \cdots$ the eigenvalues of L_u. The classical max–min formula yields

$$\begin{aligned} \lambda_n(u) &= \sup\{m(F), F \subset L^2_+, \dim F = n\}, \\ m(F) &:= \inf\{Q(f,f), f \in H^{1/2}_+,\ f \perp F, \|f\|_{L^2} = 1\}. \end{aligned}$$

Given a subspace F of L^2_+ such that $\dim F = n$, consider the subspsace

$$G(F) := \mathbb{C}1 \oplus S(F).$$

Then $G(F)$ is $(n+1)$ dimensional and

$$G(F)^\perp = S(F^\perp).$$

Consequently,

$$m(G(F)) = \inf\{Q(Sf, Sf), f \in H^{1/2}_+,\ f \perp F, \|f\|_{L^2} = 1\}$$

which, in view of (11), leads to $m(G(F)) = m(F) + 1$. From the above max–min formula, we infer

$$\lambda_{n+1}(u) \geq \sup\{m(G(F)), F \subset L^2_+, \dim F = n\} = \lambda_n(u) + 1.$$

□

For every $n \geq 1$, we define the n-th gap of L_u as

$$\gamma_n(u) = \lambda_n(u) - \lambda_{n-1}(u) - 1 \geq 0.$$

As we will see later, these quantities turn out to be the crucial action variables for the Benjamin–Ono dynamics.

2.4. **The generating functional.** Reformulating identity (11) yields

$$S^* L_u S = L_u + \mathrm{Id}\,. \tag{12}$$

Equation (12) implies the following identity for the resolvent of L_u,

$$S^*(L_u + \lambda\,\mathrm{Id})^{-1} S - (L_u + (\lambda+1)\,\mathrm{Id})^{-1} = \frac{\langle .\, | S^* w_\lambda\rangle}{\langle w_\lambda | 1\rangle}\, S^* w_\lambda \tag{13}$$

where $w_\lambda := (L_u + \lambda\,\mathrm{Id})^{-1} 1$ and where the vector 1 arises because of

$$SS^* = \mathrm{Id} - \langle .\, | 1\rangle 1.$$

Here λ is a large enough real parameter. The quantity

$$\mathcal{H}_\lambda(u) := \langle w_\lambda | 1\rangle = \langle (L_u + \lambda\,\mathrm{Id})^{-1} 1 | 1\rangle \tag{14}$$

is called the generating functional and will play a fundamental role in what follows. Taking the traces of both sides of (13), we obtain

$$-\frac{d}{d\lambda} \log \mathcal{H}_\lambda(u) = \frac{1}{\lambda_0(u) + \lambda} - \sum_{n=1}^{\infty} \frac{\gamma_n}{(\lambda_n(u) + \lambda)(\lambda_{n-1}(u) + 1 + \lambda)}.$$

Integrating from λ to $+\infty$, we infer the following factorization formula,

$$\mathcal{H}_\lambda(u) = \frac{1}{(\lambda_0(u) + \lambda)} \prod_{n=1}^{\infty} \left(1 - \frac{\gamma_n(u)}{\lambda_n(u) + \lambda}\right) \tag{15}$$

which, by analytic continuation, is valid for every complex number λ different from $-\lambda_n(u), n \geq 0$.

Formula (15) has many consequences. By inspecting the behavior of both sides as $\lambda \to +\infty$, one first infers that the series of $\gamma_n(u)$ is convergent and

$$\lambda_0(u) + \sum_{n=1}^{\infty} \gamma_n(u) = -\langle u | 1\rangle = 0.$$

Consequently,

$$\forall n \geq 0,\ \lambda_n(u) = n - \sum_{k=n+1}^{\infty} \gamma_k(u). \tag{16}$$

Then, if $u \in L^2$, one can push the expansion one step further and obtain

$$\frac{1}{2}\|u\|_{L^2}^2 = \|L_u(1)\|_{L^2}^2 = \sum_{n=1}^{\infty} n\gamma_n(u). \tag{17}$$

More generally, if u is smooth enough, pushing the expansion leads to the expression of

$$\mathcal{E}_j(u) := \langle L_u^{j+2}(1)|1\rangle,\ j = 0, 1, \ldots$$

in terms of the sequence $(\gamma_n(u))_{n\geq 1}$. For instance, if $u \in H^{1/2}_{r,0}$,

$$\mathcal{E}_1(u) = \frac{1}{2}\langle |D|u, u\rangle - \frac{1}{3}\langle u^3|1\rangle = \sum_{n=1}^{\infty} n^2\gamma_n(u) - \sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\gamma_k(u)\right)^2. \tag{18}$$

The second set of consequences of (15) relies on the identification of the residue of the meromorphic function $\lambda \mapsto \mathcal{H}_\lambda(u)$ at the pole $\lambda = -\lambda_n(u)$. Denoting by $(f_n)_{n\geq 1}$ any orthonormal basis of L^2_+ made of eigenfunctions of L_u, we have

$$\mathcal{H}_\lambda = \sum_{n=1}^{\infty} \frac{|\langle 1|f_n\rangle|^2}{\lambda + \lambda_n}$$

and consequently, comparing with (15),

$$|\langle 1|f_0\rangle|^2 = \kappa_0,\ |\langle 1|f_n\rangle|^2 = \kappa_n\gamma_n,\ n \geq 1, \tag{19}$$

where

$$\begin{aligned}
\kappa_0(u) &:= \prod_{n=1}^{\infty}\left(1 - \frac{\gamma_n(u)}{\lambda_n(u) - \lambda_0(u)}\right), \\
\kappa_n(u) &:= \frac{1}{(\lambda_n(u) - \lambda_0(u))}\prod_{1\leq p\neq n}^{\infty}\left(1 - \frac{\gamma_p(u)}{\lambda_p(u) - \lambda_n(u)}\right),\ n \geq 1.
\end{aligned}$$

Note that $\kappa_n(u) > 0$ for every $n \geq 0$. Consequently, (19) implies that

$$\langle 1, f_0\rangle \neq 0 \text{ and } \forall n \geq 1,\ \langle 1|f_n\rangle = 0 \iff \gamma_n = 0. \tag{20}$$

2.5. **The definition of Φ.** The definition of Φ relies on an appropriate choice of an orthonormal basis (f_n) of eigenfunctions of L_u. The starting point is a more precise form of (12),

$$L_uSf = S(L_uf + f) - \langle uSf|1\rangle. \tag{21}$$

Applying this identity to $f = f_n$ and taking the inner product with f_{n+1}, we infer

$$\gamma_{n+1}\langle f_{n+1}|Sf_n\rangle = \langle f_{n+1}|1\rangle\langle 1|uSf_n\rangle.$$

Consequently, if $\langle f_{n+1}|Sf_n\rangle = 0$, then

- Either $\langle 1|uSf_n\rangle = 0$, so that, coming back to (21), Sf_n is an eigenfunction of L_u with the eigenvalue $\lambda_n + 1$, which must be λ_{n+1} because of Lemma 1.
- Or $\langle f_{n+1}|1\rangle = 0$, in which case $\gamma_{n+1} = 0$ from (20), and $f_{n+1} = Sg_n$ where, from (21), $L_u g_n = \lambda_n f_n$.

In both cases, this implies that f_{n+1} is collinear to Sf_n, which contradicts $\langle f_{n+1}|Sf_n\rangle = 0$.
We conclude that, for every $n \geq 0$, $\langle f_{n+1}, Sf_n\rangle \neq 0$. By an easy induction, we infer

Lemma 2. *For every u, there exists a unique choice of the orthonormal basis $(f_n)_{n\geq 0}$ of L^2_+ such that*

$$\langle 1|f_0\rangle > 0 \text{ and } \forall n \geq 0, L_u f_n = \lambda_n f_n,\ \langle f_{n+1}, Sf_n\rangle > 0.$$

We denote by $(f_n[u])_{n\geq 0}$ the basis provided by Lemma 2. For every $n \geq 1$, we define

$$\zeta_n(u) := \frac{\langle 1|f_n[u]\rangle}{\sqrt{\kappa_n(u)}}. \tag{22}$$

Because

$$L_{u(.+\tau)}[f(.+\tau)] = (L_u f)(.+\tau),$$

the definition of $f_n[u]$ according to Lemma 2 combined with (22) easily imply property (iii) of Theorem 4.
Furthermore, in view of(19), we have

$$|\zeta_n(u)|^2 = \gamma_n(u)$$

so that, if $u \in H^0_{r,0}$, (17) becomes the claimed nonlinear version (5) of the Parseval identity, which implies that Φ sends $H^0_{r,0}$ into $h^{1/2}_+$, and that B is bounded in $H^0_{r,0}$ if and only if $\Phi(B)$ is bounded in $h^{1/2}_+$.
Proving a similar property for the map $\Phi : H^s_{r,0} \to h^{s+1/2}_+$ requires significantly more work. Let us sketch the proof in the case $s \in (-1/2, 0)$. Firstly, the product estimate (9) implies that L_u is uniformly bounded from below if u belongs to a bounded subset of H^s. Consequently, in view of (16), we have

$$\sum_{n=1}^{\infty} \gamma_n(u) \leq C_s(R)$$

if $\|u\|_{H^s} \leq R$. Then the main step consists in using the information about the domain of L_u and some complex interpolation to conclude that the linear mapping

$$K_{u,t} : f \in H^t_+ \to \langle (f|f_n[u]\rangle)_{n\geq 0} \in h^t_{\geq 0}$$

is bounded for $t \in [-1-s, 1+s]$, with a uniformly bounded norm if $\|u\|_{H^s} \le R$. Applying this to $t = s$ and observing that

$$\forall n \ge 1,\ \langle \Pi u | f_n[u] \rangle = -\lambda_n(u)\kappa_n(u)^{1/2}\zeta_n(u)$$

one eventually concludes that, if $\|u\|_{H^s} \le R$,

$$\sum_{n=1}^{\infty} n^{1+2s} |\zeta_n(u)|^2 \le C_s(R).$$

As for the reverse inequality, it is based on

$$\|f\|_{L^2}^2 \le \langle (L_u - \lambda_0(u) + 1) f | f \rangle \le M_u \|f\|_{H^{1/2}}^2,\ M_u := C_s\|u\|_{H^s} + 2 - \lambda_0(u),$$

which again comes from (9). Duality and interpolation lead to

$$\|f\|_{H^{-\theta/2}}^2 \le M_u^\theta \, \|(L_u - \lambda_0(u) + 1)^{-\theta/2} f\|_{L^2}^2$$

for every $\theta \in [0,1]$. Applying this estimate to $f = \Pi u$ with $\theta = -2s$ eventually yields

$$\|u\|_{H^s} \le N_s\left(\sum_{n=1}^{\infty} n^{1+2s} |\zeta_n(u)|^2\right)$$

for some nondecreasing function N_s.

Let us come briefly to continuity properties. The weak sequential continuity of $\Phi : H^s_{r,0} \to h_+^{s+1/2}$ is a consequence of the above estimates and of the weak sequential continuity of the maps $u \mapsto \mathcal{H}_\lambda(u)$ and $u \in H^s_{r,0} \mapsto f_n[u] \in H^{1+s}$ for each n. The strong continuity requires a little more work, using again interpolation arguments and the operator $K_{u,s}$ above. We refer to [14] for detailed proofs.

2.6. **The inverse spectral formula.** The next crucial step is the injectivity of Φ, which is based on some inverse spectral formula. The starting point is the definition of Fourier coefficients in terms of S,

$$\forall k \ge 0,\ \widehat{u}(k) = \langle (S^*)^k \Pi u | 1 \rangle. \tag{23}$$

Expressing the inner product in the right hand side with components in the basis $(f_n[u])_{n\ge 0}$, one obtains

$$\forall k \ge 0,\ \widehat{u}(k) = \langle M^k X | Y \rangle, \tag{24}$$

where $X := (\langle \Pi u | f_n[u] \rangle_{n\ge 0}$, $Y := (\langle 1 | f_n[u])_{n\ge 0}$ and

$$M_{np} := \langle S^* f_p[u] | f_n[u] \rangle.$$

A careful calculation [13] shows that the infinite sequences X, Y and the infinite matrix M only depend on the sequence $(\zeta_n(u))_{n\ge 0}$. This immediately implies that Φ is one to one.

2.7. **The case of finite gap potentials.** In order to complete the proof of Theorem 4, we need to establish that Φ is surjective and reduces the Benjamin–Ono evolution according to property (ii). This will be a consequence of the action of Φ on the symplectic form

$$\omega(h_1, h_2) = \langle h_1 | \partial_x^{-1} h_2 \rangle$$

defined on $H^{-1/2}_{r,0}$.
Then the Benjamin–Ono equation formally reads as the Hamiltonian evolution associated to the densely defined energy

$$\mathcal{E}_1(u) = \frac{1}{2}\langle |D|u|u\rangle - \frac{1}{3}\langle u^3|1\rangle = \sum_{n=1}^{\infty}\left[n^2|\zeta_n(u)|^2 - \left(\sum_{k=n}^{\infty}|\zeta_k(u)|^2\right)^2\right]$$

The crucial point will be that Φ is a symplectic map in the following sense,

$$\Phi_*\omega = i\sum_{n=1}^{\infty} d\zeta_n \wedge d\overline{\zeta}_n.$$

In fact, we are going to prove this identity on a sequence of finite dimensional symplectic submanifolds whose union is dense in $H^s_{r,0}$ for every $s > -1/2$.

Definition 1. *Let N be a positive integer. A fonction $u \in \bigcup_{s>-1/2} H^s_{r,0}$ is a N-gap potential if it satisfies*

$$\gamma_N(u) > 0 \text{ and } \forall n > N, \gamma_n(u) = 0.$$

We denote by $\mathcal{U}_N$ the set of N-gap potentials. For convenience, we set $\mathcal{U}_0 := \{0\}$.

Theorem 5. *The set $\mathcal{U}_N$ consists of functions of the form*

$$u(x) = \sum_{j=1}^{N}\left(\frac{1-r_j^2}{1-2r_j\cos(x+\alpha_j)+r_j^2} - 1\right) \tag{25}$$

where $r_j \in (0,1), \alpha_j \in \mathbb{T}$ are arbitrary. Furthermore, endowed with ω, $\mathcal{U}_N$ is a symplectic manifold of dimension $2N$, and

$$\Phi : u \in \mathcal{U}_N \longrightarrow (\zeta_n(u))_{1\le n\le N} \in \mathbb{C}^{N-1}\times\mathbb{C}^*$$

is a diffeomorphism such that

$$\Phi_*\omega = i\sum_{n=1}^{N} d\zeta_n \wedge d\overline{\zeta}_n.$$

For the proof, see [13]. Identity (25) in Theorem 5 identifies N–gap potentials to sums of N Poisson kernels — up to the additive constant N, in order to guarantee mean zero. An important ingredient in this identification is the inverse spectral formula derived in the previous subsection, which, for a N-gap potential, can be rewritten as

$$\sum_{k=1}^{\infty} \widehat{u}(k) z^k = \langle (\mathrm{Id} - zM_N)^{-1} X_N | Y_N \rangle_{\mathbb{C}^{N+1} \times \mathbb{C}^{N+1}}, \ |z| < 1,$$

where M_N is a $(N+1) \times (N+1)$ matrix the last row of which is identically zero, so that

$$Q(z) := \det(\mathrm{Id} - zM_N)$$

is a polynomial of degree N with no zeroes in the closed unit disc. Eventually one gets

$$\sum_{k=1}^{\infty} \widehat{u}(k) z^k = -z \frac{Q'(z)}{Q(z)}$$

which is equivalent to (25). The coefficients of Q provide us with a global system of complex coordinates, which makes $\mathcal{U}_N$ isomorphic to a Kähler manifold.

The symplectic properties of Φ are equivalent to the following Poisson bracket identities,

$$\{\zeta_n, \zeta_p\} = 0, \ \{\zeta_n, \overline{\zeta}_p\} = -i\delta_{np}.$$

A crucial step in the proof of these identities is the following Lax pair identity along the Hamiltonian evolution with energy $\mathcal{H}_\lambda$ for every number $\lambda > 0$ large enough,

$$\frac{d}{dt} L_{u(t)} = [B^\lambda(t), L_{u(t)}],$$

where

$$B^\lambda(t) := iT_{w_\lambda(t)} T_{\overline{w}_\lambda(t)}, \ w_\lambda := (L_u + \lambda \, \mathrm{Id})^{-1} 1.$$

2.8. **Completing the proof of Theorem 4.** Consider $\zeta := (\zeta_n)_{n \geq 1} \in h_+^{s+1/2}$ with $s > -1/2$. If $\zeta_n = 0$ for n large enough, we already know from Theorem 5 that $\zeta = \Phi(u)$ for some finite gap potential u. Otherwise, for every $N \geq 1$ such that $\zeta_N \neq 0$, Theorem 5 provides us with $u_N \in \mathcal{U}_N$ such that

$$\zeta_n(u_N) = \zeta_n, \ 1 \leq n \leq N.$$

As $N \to +\infty$, u_N is bounded in $H^s_{r,0}$, hence, up to a subsequence, weakly converges to some $u \in H^s_{r,0}$. By the sequential weak continuity of Φ, we infer $\Phi(u) = \zeta$. Hence $\Phi : H^s_{r,0} \to h_+^{s+1/2}$ is surjective.

As for property (ii), we first prove it if u is a N–gap potential. Since $\mathcal{E}_1$ is a function of $\gamma_n = |\zeta_n|^2, n = 1, \dots, N$, we infer from Theorem 5 that the corresponding Hamiltonian evolution can be written in the coordinates ζ_n,

$$\dot{\zeta}_n = i \frac{\partial \mathcal{E}_1}{\partial \gamma_n} \zeta_n, \ n = 1, \dots, N.$$

In view of the expression of $\mathcal{E}_1$ on $\mathcal{U}_N$,

$$\mathcal{E}_1(u) = \sum_{n=1}^{N} \left[n^2 \gamma_n(u) - \left(\sum_{k=n}^{N} \gamma_k(u) \right)^2 \right],$$

we infer property (ii) for finite gap potentials. The case of H^s solutions for s large enough follows from a density argument and from the continuity properties of the flow $S(t)$ on H^s.

3. Sketch of the proof of Theorems 1, 2, 3

3.1. **Wellposedness, traveling waves and almost periodicity.** Using Theorem 4, it is easy to extend continuously $S(t)$ to $H^s_{r,0}$ for every $s > -1/2$. Indeed, by applying Φ, this is equivalent to extend continuously to $h_+^{s+1/2}$ the mapping

$$(\zeta_n)_{n\geq 1} \longmapsto (\zeta_n \, \mathrm{e}^{it\omega_n})_{n\geq 1}$$

where

$$\omega_n := n^2 - 2 \sum_{k=1}^{\infty} \min(k,n) |\zeta_k|^2 .$$

Such an extension is trivial. The case of a solution with a non zero mean follows after applying a transformation (4).
As for traveling waves, in view of property (iii) and of property (ii) extended to $H^s_{r,0}$ for every $s > -1/2$, U is a traveling wave profile with velocity c if and only if

$$\forall n \geq 1, \ \forall t \in \mathbb{R}, \ \zeta_n(U) \mathrm{e}^{-inct} = \zeta_n(U) \mathrm{e}^{it\omega_n(U)} .$$

Since, for $U \neq 0$, the sequence $n \mapsto \omega_n(U)/n$ is increasing, this imposes $\zeta_n(U) = 0$ for all n except one value N. Applying the inverse spectral transform then leads to (3). Orbital stability of such traveling waves is then easily proved in coordinates (ζ_n).
Almost periodicity of the trajectories easily follows from the Bochner criterion and the formula $S(t)u = \Phi^{-1}\left[\zeta(u, t\Omega)\right]$, with

$$\Omega := (\omega_n(u))_{n\geq 1}, \ \zeta(u,\theta) := \left(\zeta_n(u) \, \mathrm{e}^{i\theta_n}\right)_{n\geq 1}, \ \theta = (\theta_n)_{n\geq 1} \in (\mathbb{R}/2\pi\mathbb{Z})^{\infty} .$$

3.2. **Illposedness on $H^{-1/2}$.** Finally, let us discuss the proof of the last part of Theorem 1. We consider potentials of the form

$$u(x) = v(\mathrm{e}^{ix}) + \overline{v(\mathrm{e}^{ix})},$$

where v is the holomorphic function defined in the unit disc by

$$v(z) = \frac{\varepsilon q z}{1 - qz}, \quad 0 < \varepsilon < q < 1, \qquad |z| < 1.$$

We are interested in the asymptotics $\varepsilon \to 0, q \to 1$. Note that

$$\|u\|^2_{H^{-1/2}} = -2\varepsilon^2 \log(1 - q^2).$$

On the other hand, the eigenvalue equation $L_u f = \lambda f$ reads as a differential equation in the complex domain,

$$zf'(z) - \left(\frac{\varepsilon qz}{1-qz} + \frac{\varepsilon q}{z-q} - \mu\right) f(z) = f(q)\frac{\varepsilon q}{q-z}, \qquad |z| < 1,$$

where we have set $\mu := -\lambda$, and f is holomorphic in the unit disc. A careful study of this first order linear differential equation around the singular points $z = 0$ and $z = q$ leads to a characterization of negative eigenvalues through the following equation for $\mu > 0$,

$$F(\mu,\varepsilon,q) := \int_0^q \frac{t^{\varepsilon+\mu}(1-qt)^\varepsilon}{(q-t)^\varepsilon}\left(\frac{\mu}{t} - \frac{\varepsilon q}{1-qt}\right) dt = 0.$$

From this equation, one infers the following elementary properties.

- If $F(\mu,\varepsilon,q) = 0$, then $\partial_\mu F(\mu,\varepsilon,q) > 0$.
- For every $\mu > 0$, $F(\mu,\varepsilon,q) \to -\infty$ if $\varepsilon \log(1-q^2) \to -\infty$.

From formula (16) for $n = 0$, we know that L_u has at least one negative eigenvalue. Using the first property above, we know that this eigenvalue is unique. Furthermore, in view of the second property above, it is possible to choose two sequences $\varepsilon_k \to 0, q_k \to 1$ such that

$$\varepsilon_k^2 \log(1-q_k^2) \to 0,\ \ \varepsilon_k \log(1-q_k^2) \to -\infty,$$

and such that the corresponding unique solution $\mu_k > 0$ of the equation

$$F(\mu_k, \varepsilon_k, q_k) = 0$$

satisfies $\mu_k \to +\infty$. In other words, if we denote by $u^{(k)}$ the corresponding potential, $u^{(k)}$ tends to 0 in $H^{-1/2}$ and $L_{u^{(k)}}$ has only one negative eigenvalue $\lambda_0(u^{(k)}) = -\mu_k$, which moreover tends to $-\infty$. The inverse spectral formula (24) and property (ii) of Theorem 4 then yield

$$\langle S(t)u^{(k)}|\mathrm{e}^{ix}\rangle = a_k\,\mathrm{e}^{it(1-2\mu_k)} + \sum_{n=1}^{\infty} b_{n,k}\,\mathrm{e}^{it(1+2\lambda_{n,k})}$$

with $\lambda_{n,k} \geq 0$ and

$$|a_k| \to \sqrt{2},\ \sup_k \sum_{n=1}^{\infty} |b_{n,k}| < \infty.$$

Consequently, the function $t \mapsto \langle S(t)u^{(k)}|\mathrm{e}^{ix}\rangle$ is uniformly bounded, and, given an interval I of positive length,

$$\left|\int_I \langle S(t)u^{(k)}|\mathrm{e}^{ix}\rangle\,\mathrm{e}^{it(2\mu_k-1)}\,dt\right| \to \sqrt{2}|I|,$$

which contradicts $\langle S(t)u^{(k)}|\mathrm{e}^{ix}\rangle \to 0$ on I, in view of the dominated convergence theorem.

4. Some perspectives

Let us mention some topics in the continuation of our results.

4.1. **The Benjamin–Ono hierarchy.** As we already noticed, the Benjamin–Ono equation is the Hamiltonian evolution associated to the energy $\mathcal{E}_1$. For every $j \geq 2$, it is possible to study the Hamiltonian evolution associated to the energy $\mathcal{E}_j$ by using the same nonlinear Fourier transform Φ. In [11], the third-order equation, which corresponds to $\mathcal{E}_2$, is studied in detail. In particular, though the critical scaling exponent is still $s = -1/2$, wellposedness is proved to hold in H^s if and only if $s \geq 0$. Furthermore, new traveling waves are found for this equation, which turn out to be orbitally unstable.

4.2. **More about the nonlinear Fourier transform.** A natural continuation of Theorem 4 would be to prove that Φ is a real analytic diffeomorphism. This would be useful for studying Hamiltonian perturbations of the Benjamin–Ono equations by KAM methods. A first result in this direction is [15], which proves that the moment map $u \mapsto (\gamma_n(u))_{n\geq 1}$ is real analytic from $H^s_{r,0}$ to the ℓ^1 space with weight n^{1+2s} for $s > -1/2$.

4.3. **The damped Benjamin–Ono equation.** Our nonlinear Fourier transform was also recently used in [12] to find new Lyapunov functionals and to study the long time behaviour of the following weakly damped Benjamin–Ono equation,

$$\partial_t u + \alpha(\langle u|\cos\rangle \cos x + \langle u|\sin\rangle \sin x) = H\partial_x^2 u - \partial_x(u^2),$$

where $\alpha > 0$.

4.4. **The quantum Benjamin–Ono equation.** Our nonlinear Fourier transform was recently used in [21] for establishing Bohr–Sommerfeld conditions for the quantum Benjamin–Ono equation. We refer to [21] for more references on this topic.

4.5. **The Benjamin–Ono equation on the line.** As we mentioned in the beginning of this paper, the Benjamin–Ono equation (1) can be studied on the line as well. Though it satisfies the same Lax pair structure, the inverse spectral theory is still to be completed. Let us mention the special case of small and sufficiently decaying data, for which the inverse scattering transform was performed in [8]. Some general results on the spectral properties of the Lax operator are available in [33], [34]. More recently, the nonlinear Fourier transform for multisolitons — the analogous of finite gap potentials on the line — was constructed in [26].

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