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THE CUBIC SZEGŐ FLOW AT LOW REGULARITY

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ABSTRACT. We prove that the cubic Szegő equation is well posed on the space BMO$^+$ of functions of bounded mean oscillation in the Hardy class of the disc, and we establish the Hölder regularity of this flow in the $L^2$ distance. We also show that the Cauchy problem is illposed on the corresponding $L^\infty$ space.

1. Introduction

This paper is devoted to low regularity solutions of the cubic Szegő equation on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$,

\begin{equation}
  i\partial_t u = \Pi(|u|^2 u)
\end{equation}

where $\Pi : L^2(\mathbb{T}) \to L^2_+(\mathbb{T})$ denotes the orthogonal projector onto the closed subspace $L^2_+(\mathbb{T})$ of $L^2(\mathbb{T})$ defined by the cancellation of all negative Fourier modes,

\[ \forall k < 0, \quad \hat{u}(k) = 0. \]

Recall that $L^2_+(\mathbb{T})$ can be identified to the Hardy space $H^2(\mathbb{D})$ consisting of holomorphic functions $u$ on the unit disc such that

\[
  \sup_{r<1} \int_0^{2\pi} |u(re^{ix})|^2 \, dx < \infty.
\]

In the sequel, we shall make use of this identification freely.

Equation (1) was introduced by S. Grellier and the first author in [5], where a flow on $H^s_+(\mathbb{T}) := H^s(\mathbb{T}) \cap L^2_+(\mathbb{T})$, $s \geq 1/2$, was defined, and where a Lax pair structure was discovered. In [8], this equation was identified as the time averaged effective system to the half wave equation on $\mathbb{T}$. In [6], more precise integrability properties were established, while in [7] an explicit formula for $H^s$ solutions was derived. Finally, a general nonlinear Fourier transform was constructed in [9], where almost periodicity of solutions in $H^{1/2}_+$ and growth of higher Sobolev

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norms were proved. Furthermore, analyticity of solutions was studied in [10].

Since $\Pi$ is a pseudodifferential operator of order 0, it is natural to ask about solving Equation (1) for initial data with low regularity. For instance, the ordinary differential equation
\begin{equation}
    i \partial_t u = |u|^2 u
\end{equation}
is wellposed on $L^\infty(\mathbb{T})$, with the explicit formula
\[ u(t, x) = e^{-it|u(0,x)|^2} u(0,x) . \]
The purpose of this paper is to investigate how this property is modified by the action of the pseudodifferential operator $\Pi$. It is well known that $\Pi$ is not bounded on $L^\infty(\mathbb{T})$. The space
\[ BMO_+(\mathbb{T}) = \{ \Pi(b), b \in L^\infty(\mathbb{T}) \} \]
was identified by Fefferman [3] as the intersection of $L^2_+(\mathbb{T})$ with the space $BMO(\mathbb{T})$ of functions of bounded mean oscillation introduced by John and Nirenberg, see [13], [4], as the space of functions $f \in L^1(\mathbb{T})$ such that
\begin{equation}
    \sup_I \frac{1}{|I|} \int_I |f(x) - \langle f \rangle_I| \, dx < +\infty , \quad \langle f \rangle_I := \frac{1}{|I|} \int_I f(x) \, dx ,
\end{equation}
where the supremum above is taken on all intervals $I \subset \mathbb{T}$. The space $BMO_+$ is also the dual of
\[ L^1_+(\mathbb{T}) = \{ h \in L^1(\mathbb{T}) : \forall k < 0 , \hat{h}(k) = 0 \} . \]
For every $u \in BMO_+(\mathbb{T})$, we set
\[ \| u \|_{BMO} = \inf \{ \| b \|_{L^\infty}, b \in L^\infty(\mathbb{T}), \Pi(b) = u \} = \| u \|_{(L^1_+)^*} . \]

Our main result is the following.

**Theorem 1.** For every $u_0 \in BMO_+(\mathbb{T})$, there exists a unique function $u \in C^1(\mathbb{R}, L^2_+(\mathbb{T})) \cap C_{w*}(\mathbb{R}, BMO_+(\mathbb{T}))$, solution of the initial value problem
\begin{equation}
    i \partial_t u = \Pi(|u|^2 u) , \quad u(0) = u_0 .
\end{equation}
Furthermore, $\| u(t) \|_{BMO} = \| u_0 \|_{BMO}$. Moreover, if $u, v$ are two BMO solutions of (1) satisfying
\[ \| u(0) \|_{BMO} + \| v(0) \|_{BMO} \leq M , \]
there exists a constant $K$, depending only on $M$, such that, for every $t \in \mathbb{R}$,
\begin{equation}
    \| u(t) - v(t) \|_{L^2} \leq K \| u(0) - v(0) \|_{L^2} , \quad \alpha(t) := e^{-K|t|} .
\end{equation}

Next we come to propagation of Sobolev regularity. In the low regularity case, it is only partially obtained, as a consequence of the stability estimate (5).
Corollary 1. Let $u$ be a BMO solution of the cubic Szegő equation, as given by Theorem 1. Assume $u(0) = u_0 \in H^s$ for some $s > 0$. Then, if $s \geq 1/2$, $u(t) \in H^s(T)$ for every $t \in \mathbb{R}$. In the case $0 < s < 1/2$, there exists $K > 0$, depending only on a bound of $\|u_0\|_{\text{BMO}}$, such that

$$\forall t \in \mathbb{R}, u(t) \in H^s(T), \quad s(t) := \frac{se^{-K|t|}}{1 - 2s + 2se^{-K|t|}}.$$ 

Remark 1.

- We do not know whether the above exponent $s(t)$ is optimal or not. If it is optimal, such a loss of regularity could be compared to the one established by Bahouri and Chemin in Theorem 1.3 of [1] for the bidimensional incompressible Euler flow with bounded vorticity.
- The above corollary has a local version, which will be established in the forthcoming paper [11].

In the beginning of this note, we have seen that the ordinary differential equation (2) is well posed on $L^\infty(T)$. In contrast, using the John–Nirenberg definition (3), it is easy to prove that this equation is not well posed on $\text{BMO}(T)$. Indeed, though $u_0(x) = \log|\sin x|$ belongs to $\text{BMO}(T)$, one can check that, for every $t \neq 0$, the function

$$u(t, x) = (\log|\sin x|)e^{-it(\log|\sin x|)^2}$$

does not belong to $\text{BMO}(T)$, because its average on $[\varepsilon, 2\varepsilon]$ is bounded as $\varepsilon$ tends to 0. Somewhat symmetrically, the next result shows that the Szegő equation is ill posed on $L^\infty$. We denote by $C_+(T) = C(T) \cap L^2_+(T)$ the Banach space of continuous functions on the circle with nonnegative Fourier modes.

Theorem 2. There exists a dense $G_\delta$ subset $\mathcal{G}$ of $C_+(T)$ such that, for every $u_0 \in \mathcal{G}$, the solution $u$ of (4) satisfies

$$\forall T > 0, u \notin L^\infty([0, T] \times \mathbb{T}).$$

The present note will give a sketch of the proof of Theorem 1, Corollary 1 and Theorem 2. An extended version with more detailed proofs and additional results is in preparation [11].

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2. Proof of Theorem 1

The proof of Theorem 1 is based on two arguments. The first one is a characterization of $\text{BMO}_+(T)$ which was established by Nehari [16] before the John–Nirenberg paper. Nehari’s result — see also Theorem 1.1 of Peller [17] — claims that, given $u \in L^2_+(T)$, the Hankel operator $\Gamma_{\hat{u}}$ defined on finitely supported sequence $x := (x_n)_{n \geq 0}$ by

$$[\Gamma_{\hat{u}}(x)]_p = \sum_{n=0}^{\infty} \hat{u}(p + n)x_n$$

Exp. n° XIV — The cubic Szegő flow at low regularity
extends as a bounded operator on $\ell^2(\mathbb{N})$ if and only if $u \in \text{BMO}_+(\mathbb{T})$, and that
\[ \|\hat{\Gamma}u\|_{\ell^2 \to \ell^2} = \|u\|_{\text{BMO}}. \]

As we will recall below, it turns out that the Lax pair discovered in [5] allows to prove that, if a $u$ is a smooth solution of (1), the operator norm $\|\hat{\Gamma}(t)\|_{\ell^2 \to \ell^2}$ is independent of $t$. This provides a BMO bound for the sequence $(u_n)$ of smooth solutions of (1) which approximates $u_0$ at $t = 0$ in $\mathcal{B}_{\text{BMO}}(\|u_0\|_{\text{BMO}}).

The second argument relies on the John–Nirenberg inequality [13], [4], which claims that $\text{BMO}_+(\mathbb{T}) \subset L^p(\mathbb{T})$ for every $p < \infty$, and that there exists a universal constant $C > 0$ such that, for every $v \in \text{BMO}_+(\mathbb{T})$, for every $p \in [1, \infty)$,
\[ \|u\|_{L^p} \leq C_p \|v\|_{\text{BMO}}. \]

This inequality will allow us to prove that the sequence $(u_n)$ is a Cauchy sequence in $C([-T, T], L^2_+(\mathbb{T}))$ for every $T < \infty$, leading to existence of solution $u$.

Let us come to the detailed proof of Theorem 1. We first recall the Lax pair structure of the cubic Szegő equation, as established in [5] and revisited in [7]. For every $u \in \text{BMO}_+(\mathbb{T})$, define the antilinear Hankel operator $H_u : L^2_+(\mathbb{T}) \to L^2_+(\mathbb{T})$
by the formula
\[ H_u(h) = \Pi(u \bar{h}) , \ h \in L^2_+(\mathbb{T}). \]

It is easy to check that $H_u$ is bounded on $L^2 +_1$, and that
\[ \hat{H}_u(h) = \hat{\Gamma}_u(\hat{h}) , \ \langle H_u(h_1), h_2 \rangle = \langle H_u(h_2), h_1 \rangle, \]
where $\langle f, g \rangle$ denotes the usual $L^2$ inner product. In particular,
\[ H^2_u \simeq \Gamma_u^{\ast} \Gamma_u \]
is a linear positive selfadjoint operator. From Nehari’s theorem, we have
\[ \|H_u\|_{L^2_+ \to L^2_+} = \|u\|_{\text{BMO}}. \]

Next we claim that, for every $a, b, c \in L^\infty_+(\mathbb{T})$,
\[ H_{\Pi(ab,c)} = T_{ab}H_c + H_aT_{bc} - H_aH_bH_c , \]
where, for every $m \in L^\infty(\mathbb{T})$, the Toeplitz operator $T_m$ is defined by
\[ T_m(h) = \Pi(mh) , \ h \in L^2_+(\mathbb{T}). \]
Indeed, given \( h \in L^2_+ (\mathbb{T}) \), we have
\[
H_{\Pi(abc)}(h) = \Pi(\Pi(abc)\overline{h}) = \Pi(abc\overline{h})
= \Pi(ab\overline{\Pi(c\overline{h})}) + \Pi(ab(I-\Pi)(c\overline{h}))
= T_{ab}H_c(h) + H_a \left( b (I-\Pi)(c\overline{h}) \right).
\]
The proof of (7) is completed by observing that
\[
\overline{b(I-\Pi)(c\overline{h})} = \Pi \left( \overline{b(I-\Pi)(c\overline{h})} \right) = T_{bc}(h) - H_bH_c(h).
\]
Now assume that \( u \) is a smooth solution to (1). Combining the equation and identity (7), we have
\[
dt H_u = -iH_{|u|^2} = -i(H_u T_{|u|^2} + T_{|u|^2} H_u - H_u^3) = [B_u, H_u],
\]
where \([B, C]\) denotes the commutator of the operators \( B, C \),
\[
B_u := -iT_{|u|^2} + \frac{i}{2} H_u^2,
\]
and where we have used the antilinearity of \( H_u \) in writing
\[
i(H_u A + AH_u) = [iA, H_u]
\]
for every linear operator \( A \). Notice that \( B_u \) is an antiselfadjoint linear operator on \( L^2_+ (\mathbb{T}) \). Solving the linear ODE
\[
\frac{dU}{dt} = B_u U , \quad U(0) = I .
\]
in the space of bounded operators on \( L^2_+ \), we get a one parameter family \( U(t) \) of unitary operators, which satisfies
\[
\forall t \in \mathbb{R} , \quad H_{u(t)} = U(t)H_{u(0)}U(t)^* .
\]
From (9) and (6), we conclude
\[
\forall t \in \mathbb{R} , \quad \|u(t)\|_{\text{BMO}} = \|u_0\|_{\text{BMO}} .
\]
We now come to the second step of the proof, for which the main point is the following stability lemma.

**Lemma 1.** Let \( u, v \) be two smooth solutions of (1), satisfying
\[
\|u_0\|_{\text{BMO}} + \|v_0\|_{\text{BMO}} \leq M .
\]
There exists a constant \( K \), depending only on \( M \), such that, for every \( t \in \mathbb{R} \),
\[
\|u(t) - v(t)\|_{L^2} \leq K \|u_0 - v_0\|_{L^2} e^{-K|t|} .
\]
**Proof.** Recall that we denote by
\[
\langle f, g \rangle = \int_0^{2\pi} f(e^{ix}) \overline{g(e^{ix})} \frac{dx}{2\pi}
\]
the inner product on $L^2(\mathbb{T})$. Set $N(t) := \|u(t) - v(t)\|_{L^2}^2$. Assume $t \geq 0$ for simplicity, and compute

$$\frac{dN}{dt} = 2\text{Im} \langle \Pi(|u|^2 u) - \Pi(|v|^2 v), u - v \rangle.$$ 

Applying the Taylor formula, we have, introducing $w_\theta := \theta u + (1 - \theta)v$ for $\theta \in [0, 1]$,

$$\Pi(|u|^2 u) - \Pi(|v|^2 v) = \int_0^1 (2T|w_\theta|^2 + H_{w_\theta})(u - v) \, d\theta.$$ 

Since $T|w_\theta|^2$ is selfadjoint, its contribution to the imaginary part of the inner product cancels, and we are left with

$$\frac{dN}{dt} = 2\int_0^1 \text{Im} \langle H_{w_\theta}(u - v), u - v \rangle \, d\theta.$$ 

Using identity (7) with $a = c = w_\theta$ and $b = 1$, we obtain

$$\frac{dN}{dt} = 2\int_0^1 \text{Im} \langle (T_{w_\theta} H_{w_\theta} + H_{w_\theta} T_{w_\theta} - H_{w_\theta} H_{w_\theta}) (u - v), u - v \rangle \, d\theta$$

$$= 4\int_0^1 \text{Im} \langle H_{w_\theta} (u - v), \overline{w_\theta} (u - v) \rangle \, d\theta + 2\int_0^1 \text{Im} \langle (w_\theta, u - v)^2 \rangle \, d\theta.$$ 

From the conservation of the BMO norm (10), we already know that $\|w_\theta\|_{\text{BMO}} \leq M$, and thus

$$\|H_{w_\theta}\|_{L_2^1 \rightarrow L_2^1} \leq M, \quad \|w_\theta\|_{L^p} \leq C M^p.$$ 

Using Hölder’s inequality, we infer, for large $p$ and for every time $t \geq 0$,

$$|\langle H_{w_\theta} (u - v), \overline{w_\theta} (u - v) \rangle| \leq M \|u - v\|_{L^2} \|w_\theta u - v|^{2/p} \|u - v|^{1-2/p} \|_{L^2}$$

$$\leq M \|u - v\|_{L^2} \|w_\theta u - v|^{2/p} \|u - v|^{1-2/p} \|_{L^{2p/p-2}}$$

$$\leq M (CM^p)^{1+2/p} \|u - v\|_{L^2}^{2/p} \|w_\theta u - v|^{2-2/p} \|_{L^{2p/p-2}} \leq \tilde{C}(M)pN^{1-1/p}.$$ 

We now choose, at a given time $t \geq 0$,

$$p = p(t) = 2 + \log(M^2/N(t)) \geq 2,$$

since, by the conservation of $L^2$ norms of $u$ and $v$,

$$N(t) \leq (\|u_0\|_{L^2} + \|v_0\|_{L^2})^2 \leq M^2.$$ 

We infer

$$\left| \frac{dN}{dt} \right| \leq K(M)N(2 + \log(M^2/N)) .$$ 

Solving this differential inequality, we obtain the lemma. \qed
Let us complete the proof of Theorem 1. Let \( u_0 \in \text{BMO}_+(T) \). Select a sequence \( (u^n_0) \) of smooth functions in \( L^2_+ \) such that
\[
\|u^n_0 - u_0\|_{L^2} \to 0, \quad \limsup \|u^n_0\|_{\text{BMO}} \leq \|u_0\|_{\text{BMO}}.
\]
For instance, one can choose
\[
u^n_0(x) = u_0(r_ne^{ix}),
\]
where \( r_n \) is any sequence of positive numbers smaller than 1 converging to 1. Denote by \( u^n \) the solution of (1) with initial datum \( u^n_0 \). Then Lemma 1 implies that \( (u^n) \) is a Cauchy sequence in \( C([-T,T],L^2_+) \) for every \( T > 0 \), hence it converges to \( u \in C(\mathbb{R},L^2_+) \). Furthermore,
\[
\|u^n(t)\|_{\text{BMO}} = \|u^n_0\|_{\text{BMO}},
\]
hence \( u_n(t) \to u(t) \) in \( L^p \) for every \( p < \infty \), locally uniformly in time. This allows to pass to the limit in Equation (1), so that \( u \) is a solution of (4), and moreover
\[
\|u(t)\|_{\text{BMO}} \leq \limsup \|u^n_0\|_{\text{BMO}} \leq \|u_0\|_{\text{BMO}}.
\]
It remains to prove uniqueness of such solutions, and the conservation of the BMO norm. For uniqueness, we observe that the proof of Lemma 1 can be easily extended to solutions \( u,v \in C(\mathbb{R},L^2_+) \cap C_{w^*}(\mathbb{R},\text{BMO}_+(T)) \). Indeed, the only technical point is to extend the identity
\[
\Pi(|w|^2h) = wH_w(h) + H_w(wh) - H_wH_1H_w(h)
\]
and the case \( w,h \in \text{BMO}_+ \). This can be easily achieved by approximation of \( w \). This leads to estimate (5). Applying this estimate to \( u_0 = v_0 \), we conclude that there exists only one solution \( u \in C(\mathbb{R},L^2_+) \cap C_{w^*}(\mathbb{R},\text{BMO}_+(T)) \) of (4).

As for the conservation of the BMO norm, it is enough to observe that, given \( T \in \mathbb{R} \), that we already have an inequality,
\[
\|u(T)\|_{\text{BMO}} \leq \|u_0\|_{\text{BMO}}.
\]
Now, precisely from what we did, the problem
\[
i\partial_t v = \Pi(|v|^2v), \quad v(0) = u(T)
\]
has only one solution \( v \in C(\mathbb{R},L^2_+) \) and locally bounded in BMO, and \( \|v(t)\|_{\text{BMO}} \leq \|v(0)\|_{\text{BMO}} \). Therefore \( v(t) = u(t + T) \), and applying the above inequality at \( t = -T \) yields \( \|u_0\|_{\text{BMO}} \leq \|u(T)\|_{\text{BMO}} \), whence the desired equality.

3. Proof of Corollary 1

In the case \( s \geq 1/2 \), Corollary 1 is just a consequence of the uniqueness of the Cauchy problem in Theorem 1 and of the wellposedness theory in \( H^s \) [5].
In the case $0 < s < 1/2$, a first idea is to combine the stability estimate (5), the invariance of the flow by translation on $\mathbb{T}$, and the following representation of the $H^s$ norm, 

$$
\|u\|_{H^s}^2 = \|u\|_{L^2}^2 + \int_{-1}^{1} \int_{\mathbb{T}} \frac{|u(x+h) - u(x)|^2}{|h|^{1+2s}} \, dx \, dh .
$$

However, this provides a result which does not take into account the conservation of the $H^{1/2}$ norm. Therefore we prefer to use the following interpolation argument, which was suggested to us by D. Tataru. Given $\lambda > 1$, one can decompose $u_0 \in H^s$ as 

$$
u_0 = u_0^{<\lambda} + u_0^{>\lambda},$$

with $\|u_0^{<\lambda}\|_{\text{BMO}} \lesssim 1$, 

$$
\|u_0^{<\lambda}\|_{H^{1/2}} \lesssim \lambda^{1-s}, \quad \|u_0^{>\lambda}\|_{L^2} \lesssim \lambda^{-s}.
$$

Then, by the conservation of the $H^{1/2}$ norm, $u^{<\lambda} := Z(u_0^{<\lambda})$ satisfies 

$$
\|u^{<\lambda}(t)\|_{H^{1/2}} \lesssim \lambda^{1-s},
$$

while the stability estimate (5) yields, with $\alpha(t) = e^{-K|t|}$ and $K = K(\|u_0\|_{\text{BMO}})$, 

$$
\|u(t) - u^{<\lambda}(t)\|_{L^2} \lesssim \|u_0 - u_0^{<\lambda}\|_{\alpha(t)} \lesssim \lambda^{-s\alpha(t)}.
$$

Therefore the the dyadic component $\Delta_k u(t)$ of $u(t)$ can be estimated, for every $\lambda > 0$, as 

$$
\|\Delta_k u(t)\|_{L^2} \lesssim 2^{-k/2} \lambda^{1-s} + \lambda^{-s\alpha(t)}.
$$

Choosing $\lambda = \lambda(k, t)$ optimally, we obtain 

$$
\|\Delta_k u(t)\|_{L^2} \lesssim 2^{-k\alpha(t)/(1-2s+2s\alpha(t))},
$$

and therefore $u(t) \in H^s(t)$ with 

$$
s(t) = \frac{se^{-\tilde{K}|t|}}{1 - 2s + 2se^{-\tilde{K}|t|}}
$$

for every $\tilde{K} > K$. This completes the proof.

4. Proof of Theorem 2

The arguments for Theorem 2 are an adaptation of a method developed by Elgindi and Masmoudi in [2], which leads to ill-posedness for the incompressible Euler equation at the $C^1$ regularity. The crucial step is the following lemma.

**Lemma 2.** Let $u_0 \in C_+(\mathbb{T})$. There exists a sequence $(u^n)$ of smooth solutions to the (1) such that 

$$
\|u^n(0) - u_0\|_{L^\infty} \to 0 ,
$$
and a sequence of times $T_n > 0$ tending to 0 such that

$$\sup_{t \in [0, T_n]} \|u^n(t)\|_{L^\infty} \to \infty .$$

Let us show how Lemma 2 implies Theorem 2. For every $u_0 \in \text{BMO}_+(\mathbb{T})$ and every $t \in \mathbb{R}$, we denote by $Z(t)(u_0)$ the value $u(t)$ at time $t$ of the solution $u := Z(u_0)$ of (4) provided by Theorem 1. For every integer $p \geq 1$, denote by $\Omega_p$ the subset of those $u_0 \in C_+(\mathbb{T})$ such that, for some $r_p \in [0, 1]$, we have

$$\sup_{t \in [0, 1/p]} \sup_{r \in [0, 1]} \sup_{x \in \mathbb{T}} |Z(t)u_0(r e^{ix})| > p .$$

We claim that $\Omega_p$ is an open subset of $C_+(\mathbb{T})$. Indeed, for every $r < 1$, the map

$$u \in L^2_+(\mathbb{T}) \to u_r \in L^\infty_+(\mathbb{T}) , \quad u_r(e^{ix}) := u(re^{ix})$$

is continuous in view of the Cauchy integral formula, and the mapping

$$u_0 \in C_+(\mathbb{T}) \mapsto Z(u_0) \in C([0, 1], L^2_+(\mathbb{T}))$$

is continuous in view of Theorem 1.

Next we claim that $\Omega_p$ is dense in $C_+(\mathbb{T})$. Given $u_0 \in C_+(\mathbb{T})$, we apply Lemma 2. The sequence provided by this lemma converges to $u_0$ in $C_+(\mathbb{T})$. Furthermore, for $n$ big enough, $T_n < 1/p$ and

$$\sup_{t \in [0, T_n]} \|u^n(t)\|_{L^\infty} > p .$$

Since, for every $f \in L^\infty_+(\mathbb{T})$,

$$\|f\|_{L^\infty} = \sup_{r < 1} \sup_{x \in \mathbb{T}} |f(re^{ix})| ,$$

we conclude that $u^n$ belongs to $\Omega_p$.

Introduce

$$\mathcal{G} = \bigcap_{p \geq 1} \Omega_p .$$

Since $C_+(\mathbb{T})$ is a Banach space, the Baire theorem shows that $\mathcal{G}$ is a dense $G_\delta$ subset of $C_+(\mathbb{T})$. Furthermore, if $u_0 \in \mathcal{G}$, we have, for every $T > 0$ and every $p \geq T^{-1}$,

$$\sup_{t \in [0, T]} \sup_{r \in [0, 1]} \sup_{x \in \mathbb{T}} |Z(t)u_0(re^{ix})| > p ,$$

hence $Z(u_0) \notin L^\infty([0, T] \times \mathbb{T})$. 

XIV–9
4.1. **Proof of Lemma 2.** We shall make use of a Banach algebra $B$ of functions on the torus, invariant by $\Pi$, included into $L^\infty$, such that

$$\|uv\|_B \leq C(\|u\|_{L^\infty} \|v\|_B + \|u\|_B \|v\|_{L^\infty}),$$

and which, roughly speaking, has the same scaling properties as $L^\infty$. An example is provided by the Besov space

$$B = B^{1/2}_2 = \{u \in L^2(\mathbb{T}) : \|u\|_B = |\hat{u}(0)| + \sum_{k=0}^\infty 2^{k/2}\|\Delta_k u\|_{L^2} < \infty\},$$

where $\Delta_k u$ denotes the usual dyadic component of $u$. Indeed, $\Pi(B) \subset B$ from the definition, the inclusion $B \subset L^\infty$ is a consequence of the standard inequality

$$\|\Delta_k u\|_{L^\infty} \lesssim 2^{k/2}\|\Delta_k u\|_{L^2},$$

and the tame estimate (11) follows from paralinearising the product $uv$. The subspace $B_+ = \Pi(B)$ of $L^\infty_+$ can also be characterised by the condition

$$[u]_{B_+} := \int_0^1 \frac{1}{\sqrt{1-r}} \left( \int_0^{2\pi} |u'(re^{ix})|^2 \, dx \right)^{1/2} \, dr < \infty,$$

where $u'$ is the holomorphic derivative of $u$, the norm $|\hat{u}(0)| + [u]_{B_+}$ being equivalent to $\|u\|_B$ on $B_+$.

We now fix $\alpha \in ]0, \infty[$ and introduce, for every $\rho \in ]0, 1[$,

$$f_\rho(z) = (1 - \rho z)^{i\alpha} = e^{i\alpha \log(1 - \rho z)}, |z| \leq 1,$$

with $\log(1 - \rho z) \in \mathbb{R} + i[-\frac{\pi}{2}, \frac{\pi}{2}]$.

**Lemma 3.** The following estimates hold as $\rho$ tends to 1,

$$\|f_\rho\|_{L^\infty} \leq C, \|f_\rho\|_B \leq C \log \frac{1}{1 - \rho},$$

and for every trigonometric polynomial $g = g(z) \in L^2_+$ with $g(1) \neq 0$,

$$\|\Pi(|f_\rho|^2 g)\|_{L^\infty} \geq c(g) \log \frac{1}{1 - \rho},$$

for some $c(g) > 0$.

**Proof.** Notice that, for $x \in \mathbb{T}$,

$$f_\rho(e^{ix}) = e^{i\alpha \log(1 + \rho^2 - 2\rho \cos x)} e^{-\alpha A_\rho(x)} = e^{-\alpha A_\rho(x)} \cdot \arctan \left( \frac{\rho \sin x}{1 - \rho \cos x} \right).$$

In particular,

$$\|f_\rho\|_{L^\infty} \leq e^{\alpha \pi/2}.$$

On the other hand,

$$f'_\rho(z) = -i\alpha \rho(1 - \rho z)^{i\alpha-1},$$
so that
\[
\int_0^{2\pi} |f_\rho'(re^{ix})|^2 \, dx \lesssim \frac{1}{1 - \rho r},
\]
and
\[
[f_\rho]_{B+} \lesssim \int_0^1 \frac{dr}{\sqrt{(1-r)(1-\rho r)}} \lesssim \log \frac{1}{1-\rho}.
\]
It remains to prove the last statement. Let \( g = g(z) \in L^2_+ \) be a trigonometric polynomial. We compute
\[
\Pi(|f_\rho|^2 g)(\rho) = \frac{1}{2\pi} \int_{-\pi}^\pi \frac{|f_\rho(e^{ix})|^2 g(e^{ix})}{1 - \rho e^{-ix}} \, dx = \frac{1}{2\pi} \int_{-\pi}^\pi \frac{e^{-2\alpha A_\rho(x)} g(e^{ix})}{1 - \rho e^{-ix}} \, dx.
\]
The above integral is uniformly bounded as \( \rho \) tends to 1, except for the contribution of a neighborhood of \( x = 0 \). Symmetrizing the integration domain, we get
\[
\Pi(|f_\rho|^2 g)(\rho) = \frac{1}{2\pi} \int_0^\pi \frac{h_\rho(x) + h_\rho(-x)}{(1 - \rho)^2 + 2\rho(1 - \cos x)} \frac{dx}{2\pi},
\]
with
\[
h(x) := (1 - \rho e^{ix}) e^{-2\alpha A_\rho(x)} g(e^{ix}).
\]
Expanding \( e^{ix} \) near \( x = 0 \), we obtain,
\[
\Pi(|f_\rho|^2 g)(\rho) = O(1) + \frac{i\rho}{2\pi} g(1) \int_0^\pi \frac{x (e^{2\alpha A_\rho(x)} - e^{-2\alpha A_\rho(x)})}{(1 - \rho)^2 + 2\rho(1 - \cos x)} \, dx.
\]
Notice that function \( A_\rho \) is nonnegative on \([0, \pi]\) and increasing from \( x = 0 \) to \( x = \arccos \rho \sim \sqrt{2(1-\rho)} \). In particular, the integrand of the above integral is nonnegative, and we may restrict \( x \) to the domain of integration \([1 - \rho, \sqrt{1-\rho}]\), on which \( A_\rho(x) \gtrsim \frac{\pi}{4} \), so that
\[
\int_0^\pi \frac{x (e^{2\alpha A_\rho(x)} - e^{-2\alpha A_\rho(x)})}{(1 - \rho)^2 + 2\rho(1 - \cos x)} \, dx \geq c_\alpha \int_{1-\rho}^{\sqrt{1-\rho}} \frac{x}{(1 - \rho)^2 + x^2} \, dx \geq c_\alpha \log \frac{1}{1 - \rho}.
\]
This completes the proof of Lemma 3.
Next, we consider, for a given trigonometric polynomial $g = g(z) \in L^2_+$ such that $g(1) \neq 0$, the family of data
\[ u_0^{\rho, \varepsilon} = g + \varepsilon f_{\rho} . \]
Applying Lemma 3, we observe that
\[ \|u_0^{\rho, \varepsilon} - g\|_{L^\infty} = O(\varepsilon) , \quad \|u_0^{\rho, \varepsilon}\|_B \lesssim O(1) + \varepsilon \log \frac{1}{1 - \rho} . \]
Furthermore,
\[ \Pi(\|u_0^{\rho, \varepsilon}\|^2 u_0^{\rho, \varepsilon}) = \Pi(\|g\|^2 g) + \varepsilon [2\Pi(\|g\|^2 f_{\rho}) + \Pi(g^2 \overline{f}_{\rho})] + \varepsilon^2 [2\Pi(\|f_{\rho}\|^2 g) + \Pi(f_{\rho}^2 \overline{g})] + \varepsilon^3 \Pi(\|f_{\rho}\|^2 f_{\rho}) . \]

Notice that, if $h \in L^\infty_+$, \( \Pi(e^{-ix} h) = e^{-ix}(h - h(0)) \) belongs to $L^\infty_+$ with $\|\Pi(e^{-ix} h)\|_{L^\infty} \leq 2\|h\|_{L^\infty}$. Since $\Pi(\|g\|^2 f_{\rho})$ is a finite linear combination of terms of the form $e^{inx} f_{\rho}$ and $\Pi(e^{-inx} f_{\rho})$ with $|n|$ not greater than the degree of $g$, we conclude that $\Pi(\|g\|^2 f_{\rho})$ is bounded in $L^\infty_+$. Similarly, $\Pi(f_{\rho}^2 \overline{g})$ is bounded in $L^\infty_+$, and so is $\Pi(g^2 \overline{f}_{\rho})$, since it is a finite trigonometric polynomial of degree not greater than twice the degree of $g$, with coefficients estimated by the supremum of Fourier coefficients of $f_{\rho}$. Finally, applying (11) and Lemma 3,
\[ \|\Pi(\|f_{\rho}\|^2 f_{\rho})\|_{L^\infty} \lesssim \|\Pi(\|f_{\rho}\|^2 f_{\rho})\|_B \lesssim \|f_{\rho}\|_B \lesssim \log \frac{1}{1 - \rho} . \]

This leads to
\[ \|\Pi(\|u_0^{\rho, \varepsilon}\|^2 u_0^{\rho, \varepsilon})\|_{L^\infty} \geq \varepsilon^2(c(g) - \varepsilon C(g)) \log \frac{1}{1 - \rho} - O(1) . \]
Choosing $\varepsilon$ small enough, we infer
\[ \Pi(\|u_0^{\rho, \varepsilon}\|^2 u_0^{\rho, \varepsilon})\|_{L^\infty} \geq \varepsilon^2 \tilde{c}(g) \log \frac{1}{1 - \rho} - O(1) , \quad \tilde{c}(g) > 0 . \]

Next we consider $u^{\rho, \varepsilon} = Z(u_0^{\rho, \varepsilon})$. We claim that, for every positive time $T \ll 1$, there exists $\rho = \rho(\varepsilon, T)$ such that, for $\varepsilon \ll 1$,
\[ \limsup_{\varepsilon \to 0} \sup_{t \in [0, T]} \|u^{\rho(\varepsilon, T), \varepsilon}\|_{L^\infty} = +\infty . \]
Indeed, assume by contradiction that, for some $T > 0$ and for some $M$, we have, for some $\varepsilon_0 > 0$,
\[ \sup_{\varepsilon < \varepsilon_0} \sup_{\rho < 1} \sup_{t \in [0, T]} \|u^{\rho, \varepsilon}\|_{L^\infty} \leq M . \]
Then, from the equation
\[ u^{\rho, \varepsilon}(t) = u_0^{\rho, \varepsilon} - i \int_0^t \Pi(\|u^{\rho, \varepsilon}(s)\|^2 u^{\rho, \varepsilon}(s)) \, ds \]
and using (11), we have, if \( t \in [0,T] \),

\[
\sup_{s \in [0,t]} \| u^{\rho,\varepsilon}(s) \|_B \leq \| u^{\rho,\varepsilon}_0 \|_B + CM^2 t \sup_{s \in [0,t]} \| u^{\rho,\varepsilon}(s) \|_B,
\]

so that, if \( t \leq \tilde{T}^* := \min(T, (2CM^2)^{-1}) \),

\[
(14) \quad \sup_{s \in [0,t]} \| u^{\rho,\varepsilon}(s) \|_B \leq 2 \| u^{\rho,\varepsilon}_0 \|_B \lesssim O(1) + \varepsilon \log \frac{1}{1-\rho}.
\]

Then we write the Taylor formula at second order in \( t \),

\[
u^{\rho,\varepsilon}(t) = u^{\rho,\varepsilon}_0 - it \Pi(|u^{\rho,\varepsilon}_0|^2 u^{\rho,\varepsilon}_0) + \int_0^t (t-s) \left[-2(T_{|u^{\rho,\varepsilon}(s)|^2})^2 + H_{u^{\rho,\varepsilon}(s)^2} T_{|u^{\rho,\varepsilon}(s)|^2}\right] u^{\rho,\varepsilon}(s) \, ds,
\]

so that, using again (11) and (14), for every \( t \in [0,T^*] \),

\[
\| u^{\rho,\varepsilon}(t) - u^{\rho,\varepsilon}_0 + it \Pi(|u^{\rho,\varepsilon}_0|^2 u^{\rho,\varepsilon}_0) \|_B \leq K(M) \varepsilon t^2 \log \frac{1}{1-\rho} + O(1).
\]

Using (13), we infer

\[
\forall t \in [0,T^*], \quad \| u^{\rho,\varepsilon}(t) \|_{L^\infty} \geq t \varepsilon \log \frac{1}{1-\rho}(\tilde{c}(g)\varepsilon - tK(M)) - O(1).
\]

Choosing \( t = T^{**} := \min(T^*, \varepsilon \tilde{c}(g)/2K(M)) \) and \( \rho = \rho(\varepsilon, T) \) close enough to 1, we obtain a contradiction.

Summing up, we have proved that, for every trigonometric polynomial \( g = g(z) \in L^2_+ \), such that \( g(1) \neq 0 \), there exists a sequence of data \( u^n_0 \) converging to \( g \) in \( C_+(\mathbb{T}) \), and a sequence of positive times \( T_n \) converging to 0, such that

\[
\sup_{t \in [0,T_n]} \| Z(t)u^n_0 \|_{L^\infty} \to \infty.
\]

Since any \( u_0 \in C_+(\mathbb{T}) \) can be approximated by a sequence of trigonometric polynomials \( g \in L^2_+ \) with \( g(1) \neq 0 \), this completes the proof of Lemma 2. \( \square \)

**REFERENCES**


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XIV–14