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**STABLE SELF-SIMILAR BLOW-UP DYNAMICS FOR SLIGHTLY
 L^2 -SUPERCRITICAL GENERALIZED KDV EQUATIONS**

YANG LAN

ABSTRACT. In this paper we consider the slightly L^2 -supercritical gKdV equations $\partial_t u + (u_{xx} + u|u|^{p-1})_x = 0$, with the nonlinearity $5 < p < 5 + \varepsilon$ and $0 < \varepsilon \ll 1$. We will prove the existence and stability of a blow-up dynamics with self-similar blow-up rate in the energy space H^1 and give a specific description of the formation of the singularity near the blow-up time.

1. INTRODUCTION

We consider the following gKdV equations:

$$\begin{cases} \partial_t u + (u_{xx} + u|u|^{p-1})_x = 0, & (t, x) \in [0, T) \times \mathbb{R}, \\ u(0, x) = u_0(x) \in H^1(\mathbb{R}), \end{cases} \quad (1.1)$$

with $1 \leq p < +\infty$.

This kind of problem appears in Physics, for example in the study of waves on shallow water (see [11]). These equations, with nonlinear Schrödinger equations, are considered as universal models for Hamiltonian systems in finite dimension.

From the result of C. E. Kenig, G. Ponce and L. Vega [8], (1.1) is locally well-posed in H^1 and thus for all $u_0 \in H^1$, there exists a maximal lifetime $0 < T \leq +\infty$ and a unique solution $u(t, x) \in C([0, T), H^1(\mathbb{R}))$ to (1.1). Moreover, we have: either $T = +\infty$ or $T < +\infty$ and $\lim_{t \rightarrow T} \|u_x(t)\|_{L^2} = +\infty$.

A solution with a finite maximal lifetime $0 < T < +\infty$, is called a blow-up solution. Numerical simulation for example [1] suggests that finite time blow up may occur for some initial data. But unlike the focusing nonlinear Schrödinger equations, there are no pseudo-conformal invariance or Virial's identity, which allow us to have explicit blow-up solution. Until recently, the blow dynamics of (1.1) for $p = 5$ has been established by Martel, Merle and Raphaël in a series of papers [19, 13, 14, 15, 16, 17, 18]. They prove the existence and stability of blow-up solutions to (1.1), and also give a specific description of the asymptotic behavior near the blow-up time.

In this paper, we will focus on the case with nonlinearity $5 < p < 5 + \delta$, where $\delta > 0$ is small enough. Numerical simulation in [4] suggests that there exists self-similar blow-up solution to (1.1) in this case. Here a self-similar solution is a solution of the following form:

$$u(t, x) \sim \frac{1}{(T-t)^{\frac{2}{3(p-1)}}} P\left(\frac{x}{(T-t)^{\frac{1}{3}}}\right).$$

We mention here that, in [15] Martel and Merle proved that there is no self-similar blow-up solution for the case $p = 5$ under certain conditions.

The main issue of this paper is to construct a blow up solution to (1.1) in the energy space H^1 with self-similar blow-up rate. Moreover, we will prove its stability and give a explicit description of the formation of singularity near the blow up time.

2. PRELIMINARY

In this section we will recall some special feature of the Cauchy problem (1.1).

2.1. Conservation laws. The equation has two important conservation laws, i.e. mass and energy:

$$E(u(t)) := \frac{1}{2} \int |u_x(t)|^2 - \frac{1}{p+1} \int |u(t)|^{p+1} = E_0, \quad (2.1)$$

$$M(u(t)) := \int |u(t)|^2 = M_0. \quad (2.2)$$

2.2. Scaling invariance. Let $u(t)$ be a solution to (1.1), and $\lambda > 0$ be a positive constant. Then

$$u_\lambda(t) = \frac{1}{\lambda^{\frac{2}{p-1}}} u\left(\frac{t}{\lambda^3}, \frac{x}{\lambda}\right)$$

is still a solution to (1.1). Moreover, for all $\lambda > 0$, we have

$$\|u(0)\|_{\dot{H}^{\sigma_c}} = \|u_\lambda(0)\|_{\dot{H}^{\sigma_c}}, \quad \sigma_c = \frac{1}{2} - \frac{2}{p-1}.$$

Here

$$\dot{H}^s = \left\{ f \in \mathcal{S}' \mid \|f\|_{\dot{H}^s}^2 := \int |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi < +\infty \right\}.$$

If $\sigma_c < 0$ (or equivalently $p < 5$), the problem (1.1) is called (mass) subcritical. If $\sigma_c = 0$ or $p = 5$, the problem (1.1) is called (mass) critical. While if $\sigma_c > 0$ or $p > 5$, (1.1) is called (mass) supercritical.

From the energy and mass conservation laws, in the subcritical case, H^1 solutions are always global in time and uniformly bounded in H^1 .

2.3. Soliton solution. The problem (1.1) has a special class of solution, i.e. called the soliton solutions. They are given by

$$u(t, x) = \frac{1}{\lambda_0^{\frac{2}{p-1}}} \mathcal{Q}_p\left(\frac{x - \lambda_0^{-2}t}{\lambda_0}\right),$$

where $\lambda_0 > 0$, $x_0 \in \mathbb{R}$ and \mathcal{Q}_p is the unique radial nonnegative solution with exponential decay to the following elliptic equation:

$$\mathcal{Q}_p'' - \mathcal{Q}_p + \mathcal{Q}_p |\mathcal{Q}_p|^{p-1} = 0.$$

From [24], \mathcal{Q}_p is the minimizer to the following functional:

$$J(f) := \min_{f \in H^1(\mathbb{R}), f \neq 0} \frac{\|f_x\|_{L^2}^{\frac{p-1}{2}} \|f\|_{L^2}^{\frac{p+3}{2}}}{\|f\|_{L^{p+1}}^{p+1}}.$$

A standard variation argument shows that for $p = 5$, if $\|u_0\|_{L^2} < \|\mathcal{Q}_5\|_{L^2}$, then

$$E(u_0) = \frac{1}{2} \int |\partial_x u_0|^2 - \frac{1}{6} \int |u_0|^6 \gtrsim \int |\partial_x u_0|^2.$$

Hence, the corresponding solution with initial data u_0 is global in time. In conclusion, for the critical problem, i.e. $p = 5$, a necessary condition for blow-up is that $\|u_0\|_{L^2} \geq \|\mathcal{Q}_5\|_{L^2}$.

3. MAIN RESULT

Now we can state the result of [12].

Theorem 3.1 (Existence and stability of a self-similar blow-up dynamics). *There exists a $p^* > 5$ such that for all $p \in (5, p^*)$, there exist constants $\delta(p) > 0$ and $b^*(p) > 0$ with*

$$\lim_{p \rightarrow 5} \delta(p) = 0 \quad (3.1)$$

$$0 < c_0(p-5) \leq b^*(p) \leq C_0(p-5) \quad (3.2)$$

and a nonempty open subset \mathcal{O}_p in H^1 such that the following holds. If $u_0 \in \mathcal{O}_p$, then the corresponding solution to (1.1) blows up in finite time $0 < T < +\infty$, with the following dynamics : there exist geometrical parameters $(\lambda(t), x(t)) \in \mathbb{R}_+^* \times \mathbb{R}$ and an error term $\varepsilon(t)$ such that:

$$u(t, x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} [\mathcal{Q}_p + \varepsilon(t)] \left(\frac{x - x(t)}{\lambda(t)} \right) \quad (3.3)$$

with

$$\|\varepsilon_y(t)\|_{L^2} \leq \delta(p). \quad (3.4)$$

Moreover, we have:

- (1) *The blow-up point converges at the blow-up time:*

$$x(t) \rightarrow x(T) \text{ as } t \rightarrow T, \quad (3.5)$$

- (2) *The blow-up speed is self-similar:*

$$\forall t \in [0, T), \quad (1 - \delta(p)) \sqrt[3]{3b^*(p)} \leq \frac{\lambda(t)}{\sqrt[3]{T-t}} \leq (1 + \delta(p)) \sqrt[3]{3b^*(p)}. \quad (3.6)$$

- (3) *The following convergence holds:*

$$\forall q \in [2, \frac{2}{1-2\sigma_c}), \quad u(t) \rightarrow u^* \text{ in } L^q \text{ as } t \rightarrow T. \quad (3.7)$$

- (4) *The asymptotic profile u^* displays the following singular behavior:*

$$(1 - \delta(p)) \int \mathcal{Q}_p^2 \leq \frac{1}{R^{2\sigma_c}} \int_{|x-x(T)| < R} |u^*|^2 \leq (1 + \delta(p)) \int \mathcal{Q}_p^2. \quad (3.8)$$

for R small enough. In particular, we have for all $q \geq \frac{2}{1-2\sigma_c}$:

$$u^* \notin L^q.$$

Remark 3.2. Here the meaning of $q_c = \frac{2}{1-2\sigma_c}$ is given by the following Sobolev embedding:

$$\dot{H}^{\sigma_c} \hookrightarrow L^{q_c}.$$

That is, the asymptotic profile u^* is not in the critical space \dot{H}^{σ_c} , and the convergence (3.7) only exists in subcritical Lebesgue spaces.

Remark 3.3. It is easy to see from the L^2 conservation law that $\int |u^*|^2 = \int |u_0|^2$.

Remark 3.4. The conclusion here is almost the same to the Schrödinger case in [20]. But we need a totally differential strategy, due to the different structure of these two equations. Indeed, our strategy here is very close to the one in [16] for critical gKdV. But there are some significant difference between critical and supercritical equations. For example the singular dynamics for gKdV is located around some point $x(t)$, which always goes to infinity in finite time for critical equation¹. While in this supercritical case, $x(t)$ converges to some finite point.

Remark 3.5. Theorem 3.1 is the first construction of blow-up solutions to the supercritical gKdV equations with initial data in H^1 . This is a stable blow-up dynamics instead of a single blow-up solution. So it is not like the self-similar solution constructed by H. Koch in [9], though the construction in this paper relies deeply on H. Koch's work.

4. OUTLINE OF THE PROOF

We will give in this subsection a brief insight of the proof of Theorem 3.1. We will first use the self-similar solution constructed by H. Koch in [9], to derive a finite dimensional dynamics, which fully describe the blow-up regime. Since we are considering the slightly supercritical case, it is helpful to view this equation as a perturbation of the critical equation in some sense. So we can use some critical techniques in our analysis, though they may have a totally different meaning in the supercritical case.

4.1. Derivation of the law. We are looking for a solution to (1.1) of the form:

$$u(t, x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} V_{b(t)} \left(\frac{x - x(t)}{\lambda(t)} \right), \quad (4.1)$$

and introduce the rescaled time:

$$\frac{ds}{dt} = \frac{1}{\lambda(t)^3}.$$

Then u is a solution to (1.1) if and only if V_b solves the following equation:

$$b_s \frac{\partial V_b}{\partial b} - \frac{\lambda_s}{\lambda} \Lambda V_b + (V_b'' - V_b + V_b |V_b|^{p-1})' = \left(\frac{x_s}{\lambda} - 1 \right) V_b', \quad (4.2)$$

where Λ is the scaling operator:

$$\Lambda f = \frac{2}{p-1} f + y f'.$$

Similar to the Schrödinger case, the self-similar blow-up regime of (1.1) corresponds to the following finite dimensional dynamics:

$$\frac{ds}{dt} = \frac{1}{\lambda^3}, \quad \frac{x_s}{\lambda} = 1, \quad \frac{\lambda_s}{\lambda} = -b, \quad b_s = 0, \quad (4.3)$$

which, after integrating, leads to finite time blow-up for $b(0) > 0$ with:

$$\lambda(t) = c(u_0) \sqrt[3]{T - t}.$$

¹This somehow explains why there is no self-similar blow-up solution for critical gKdV with initial data near soliton.

4.2. Self-similar profile. From the previous analysis, we can see it is very crucial to find a solution to the following ODE:

$$b\Delta v + (v'' - v + v|v|^{p-1})' = 0. \quad (4.4)$$

This has been done by H. Koch in [9]. Indeed, Koch obtained a even larger range of solutions.

Proposition 4.1 (H. Koch [9]). *There exist $p^* > 5, b^* > 0$, such that there exist two smooth maps: $\gamma(b, p) : [0, b^*) \times [5, p^*) \rightarrow \mathbb{R}$, $v(b, p, y) : [0, b^*) \times [5, p^*) \times \mathbb{R} \rightarrow \mathbb{R}$, such that the following holds:*

(1) *The self-similar equation:*

$$b((1 + \gamma(b, p))v + xv') + (v'' - v + v|v|^{p-1})' = 0, \quad (4.5)$$

$$(v(b, p, \cdot), \mathcal{Q}_p'(\cdot)) = 0, \quad v(b, p, y) > 0. \quad (4.6)$$

(2) *For all $p \in [5, p^*)$, there exists a unique $b = b(p) \in [0, b^*)$ such that:*

$$\gamma(b(p), p) = -1 + \frac{2}{p-1}, \quad b(5) = 0, \quad (4.7)$$

Moreover,

$$\left. \frac{db(p)}{dp} \right|_{p=5} = \frac{\|\mathcal{Q}\|_{L^2}^2}{\|\mathcal{Q}\|_{L^1}^2} > 0, \quad (4.8)$$

$$\left. \frac{\partial \gamma}{\partial b} \right|_{b=b(p)} = -\frac{\|\mathcal{Q}_p\|_{L^1}^2}{8\|\mathcal{Q}_p\|_{L^2}^2} + O(|p-5|) < 0, \quad (4.9)$$

$$\frac{1}{2} \int |v_y(b(p), p, y)|^2 dy - \frac{1}{p+1} \int |v(b(p), p, y)|^{p+1} dy = 0. \quad (4.10)$$

(3) $v(b, p, \cdot) \in \dot{H}^1 \cap L^{p+1}$, $v(b, p, \cdot) \notin L^2$ if $b > 0$ and $v(0, p, y) = \mathcal{Q}_p(y)$.
Moreover,

$$|\partial_y^k \partial_b^n v| \lesssim \begin{cases} e^{-\frac{1}{3b}} (1 + b^{-2/3}|1-by|)^{-1-\gamma-k} & \text{if } y > b^{-1}, \\ e^{-\frac{y}{16}} & \text{if } b^{-1} \geq y > 0, \\ |\partial_y^k \partial_b^n (b(1-by)^{-1-\gamma})| + e^y & \text{if } y \leq 0. \end{cases} \quad (4.11)$$

Remark 4.2. From Proposition 4.1, for every $p \in (5, 5 + \delta)$, $\delta > 0$ small enough, we can find a unique $b > 0$ and a function v solving (4.4). This gives us an explicit blow-up “solution” to (1.1) with self-similar blow-up rate. But this solution never belongs to the critical Sobolev space \dot{H}^{σ_c} and hence H^1 , due to a slowly decaying tail. Thus, we can’t expect any stability result of this solution. Since a stability result usually requires us to work in a Cauchy space. In this situation, a natural Cauchy space is the energy space H^1 .

But fortunately, we have the asymptotic behavior of the solution v , i.e. (4.11). Therefore we can choose a suitable approximation of v . More precisely, we fix a $p \in (5, 5 + \delta)$, and let

$$b_c = b(p) \sim p - 5 > 0, \quad Q_b(y) = \chi(b_c y)v(v, p, y), \quad (4.12)$$

where χ is a smooth function such that $\chi(y) = 1$, if $|y| < 1$, $\chi(y) = 0$, if $|y| > 2$. Then from (4.11), we know that Q_b has exponential decay on the right. Moreover, Q_b belongs to H^1 , whose H^1 norm is of size one. We will see that this approximated self-similar profile Q_b will lead to a stable self-similar blow-up dynamics.

4.3. Decomposition of the flow. Let us assume that the initial data is near the approximated profile Q_{b_c} up to scaling and translation, which is exactly the definition of the nonempty open subset \mathcal{O}_p introduced in Theorem 3.1. Then from a standard implicit function argument introduced in [19, 15, 13, 16], we can find geometrical parameters $(\lambda(t), x(t), b(t))$ and error term $\varepsilon(t)$, such that the following decomposition holds:

$$u(t, x) = \frac{1}{\lambda^{\frac{2}{p-1}}(t)} [Q_{b(t)} + \varepsilon(t)] \left(\frac{x - x(t)}{\lambda(t)} \right). \quad (4.13)$$

Moreover, the error term satisfies the following orthogonality condition:

$$(\varepsilon(t), \mathcal{Q}_p) = (\varepsilon(t), \Lambda \mathcal{Q}_p) = (\varepsilon(t), y \Lambda \mathcal{Q}_p) = 0. \quad (4.14)$$

4.4. Modulation equation. Differentiating the orthogonality condition (4.14), we will obtain the modulation equations of the parameters, which are:

$$\begin{aligned} \frac{\lambda_s}{\lambda} + b &= O(b_c^{\frac{5}{2}} + \|\varepsilon\|_{H_{\text{loc}}^1}), \\ \frac{x_s}{\lambda} - 1 &= O(b_c^{\frac{5}{2}} + \|\varepsilon\|_{H_{\text{loc}}^1}), \\ b_s + c_p(b - b_c)b_c &= O(b_c^3 + b_c \|\varepsilon\|_{H_{\text{loc}}^1}), \\ s &= \int_0^t \frac{1}{\lambda^3(\tau)} d\tau. \end{aligned} \quad (4.15)$$

Our main task here is to control $\|\varepsilon\|_{H_{\text{loc}}^1}$, which is done by a bootstrap argument. If such a control exists, we will see that (4.15) is just a small perturbation of the system (4.3), and has the same asymptotic behavior.

4.5. Local H^1 control of the error term. The key techniques in this paper are the monotonicity of energy and a dispersive control of $\|\varepsilon\|_{H_{\text{loc}}^1}$.

In supercritical case any critical or subcritical norm of the error term ε cannot be controlled, for example $\|\varepsilon\|_{L^2}$ or even $\int_{y>0} \varepsilon^2$. This contrasts with the critical case, where $\|\varepsilon\|_{L^2}$ is small. And for the same reason, we can no longer use the L^1 control of ε on the right as Martel, Merle and Raphaël did in the critical case in [16].

Fortunately, we can still control $\|\varepsilon_y\|_{L^2}$. Moreover, from the energy conservation law and a localization argument, we can obtain an even better control of the L^2 norm of ε_y on the half-line $[\kappa B, +\infty)$:

$$\int_{y>\kappa B} \varepsilon_y^2 \lesssim b_c^{\frac{55}{7}}. \quad (4.16)$$

where $\kappa > 0$ is a small universal constant, and

$$B = b_c^{-\frac{1}{20}}.$$

Together with Gagliardo-Nirenberg inequality, we can give a good control of the localized L^2 norm of ε on the right:

$$\int_{\kappa B < y < 2B^2} \varepsilon^2 \lesssim b_c^2. \quad (4.17)$$

Next, we construct a nonlinear functional:

$$\mathcal{F} = \int \left[\varepsilon_y^2 \psi_B + \varepsilon^2 \zeta_B - \frac{2}{p+1} (|\varepsilon + Q_b|^{p+1} - Q_b^{p+1} - (p+1)\varepsilon Q_b^p) \psi_B \right],$$

for well chosen functions (ψ_B, ζ_B) , which are exponentially decaying to the left and bounded on the right. A similar functional was introduced in [16] for the critical equations, but they have a totally different meaning. Here the key point in this case is that we cannot control $\int_{y>0} \varepsilon^2$. We need to find a different way to control ε on the right. However, if we choose ζ such that it is compactly supported on the right, i.e. $\text{supp } \zeta \subset (-\infty, 2B^2]$, then for $y > 0$, only localized L^2 norm of ε appears in \mathcal{F} , which can be controlled by (4.17).

The most significant technique here is the *Lyapounov monotonicity*:

$$\frac{d\mathcal{F}}{ds} + \frac{1}{B} \|\varepsilon\|_{H_{\text{loc}}^1}^2 \lesssim b_c^{\frac{7}{2}}. \quad (4.18)$$

Moreover, the leading order term of \mathcal{F} is:

$$\mathcal{F} \sim \int \left[\varepsilon_y^2 \psi_B + \varepsilon^2 \zeta_B - p\varepsilon^2 Q_b^{p-1} \psi_B \right],$$

which is coercive up to 3 bad direction. More precisely, we have:

Lemma 4.3. *There exists a $\kappa_0 > 0$ such that for all $f \in H^1$, there holds:*

$$\int f_x^2 + f^2 - pQ_p^{p-1} f^2 \geq \kappa_0 \|f\|_{H^1}^2 - \frac{1}{\kappa_0} [(f, Q_p)^2 + (f, \Lambda Q_p)^2 + (f, y\Lambda Q_p)^2].$$

From Lemma 4.3, orthogonality condition (4.14) and a standard localization argument, we have:

$$\mathcal{F} \gtrsim \|\varepsilon\|_{H_{\text{loc}}^1}^2.$$

The above analysis shows that $\|\varepsilon\|_{H_{\text{loc}}^1}$ (or equivalently \mathcal{F}) is almost decreasing with respect to $s \in [0, +\infty)$. Hence we have:

$$\|\varepsilon\|_{H_{\text{loc}}^1}^2 \lesssim b_c^{3+8\nu}, \quad (4.19)$$

where $\nu > 0$ is a small universal constant.

4.6. Nonlinear estimate of the error term. In the previous section, there are nonlinear term like

$$\int |\varepsilon|^{p+1} \zeta_B$$

evolved in the functional \mathcal{F} . So in order to estimate \mathcal{F} , a global L^{p_0} control of the error term ε is necessary, where $p_0 < p+1$. But from scaling, we cannot expect any global control of subcritical norm of ε , i.e. we must choose $p_0 > \frac{p-1}{2}$. Indeed,

$$p_0 = \frac{5}{2}$$

is enough for our analysis. To control $\|\varepsilon\|_{L^{p_0}}$, it is better to work in the original variable, i.e. we only need to find a control of

$$\tilde{u}(t, x) = \frac{1}{\lambda^{\frac{2}{p-1}}(t)} \varepsilon \left(t, \frac{x - x(t)}{\lambda(t)} \right).$$

We write down the equation of \tilde{u} :

$$\partial_t \tilde{u} + \partial_{xxx} \tilde{u} = (f(\tilde{u}))_x + Q_S,$$

where $f(\tilde{u})$ is a nonlinear term of \tilde{u} , Q_S is the singular part of the original solution. It is easy to see that Q_S can be controlled by the modulation equations (4.15). Now, using the inhomogeneous Strichartz estimates introduced by Foschi in [5], we have:

$$\|\tilde{u}\|_{L^{p_0}} \leq \frac{b_c^{\frac{13}{28}}}{\lambda^{\frac{2}{p-1}-p_0}(t)},$$

or equivalently

$$\|\varepsilon\|_{L^{p_0}} \leq b_c^{\frac{13}{28}}, \quad (4.20)$$

which is exactly the required nonlinear estimate.

4.7. End of the proof of Theorem 3.1. From the control of the error term, we can see that the parameters $(\lambda(t), x(t), b(t))$ satisfies (4.3) up to some small perturbation. So they will have the same asymptotic behavior, which leads to the self-similar blow-up dynamics. This dynamics is also stable due to the openness of the initial data set \mathcal{O}_p .

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