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## SHARP HARDY-LITTLEWOOD-SOBOLEV INEQUALITIES ON A CLASS OF H-TYPE GROUPS

AN ZHANG

ABSTRACT. This report is based on a talk given by the author in the Laurent Schwartz seminar at IHÉS, Paris, on February 16, 2016. This involves joint works with Michael Christ and Heping Liu [CLZ16a, CLZ16b, LZ15]. We review several sharp Hardy-Littlewood-Sobolev-type inequalities (HLS) on I-type groups (rank one), which is a special class of H-type groups, using the symmetrization-free method of Frank and Lieb, who proved the sharp HLS on the Heisenberg group in a seminal paper [FL12b]. We give the sharp HLS both on the compact and noncompact pictures. The “unique” extremal function, as expected, can only be *constant* function on the sphere. Their dual form, a sharp conformally invariant inequality involving an intertwining operator (“fractional subLaplacian”), and the right endpoint case, a Log-Sobolev inequality, are also obtained. Besides, some stability and dual type improvements are also discussed. A positivity-type restriction on the singular exponent is required in the cases with centres of high dimensions, which bring extra difficulty. The conformal symmetry of the inequalities, zero center-mass technique, estimates involving meticulous computation of eigenvalues of singular kernels, compactness and local stability play a critical role in the argument.

### 1. INTRODUCTION

Sharp constants and extremal functions, called *extremizers*, for important inequalities have been studied since many years ago. It has been a general hot topic in analysis, geometry, probability, PDE and quantum field theory. They play an important role because they almost always contain or reveal profound geometric and probabilistic information of the underlying space, manifold or group. Vast beautiful literature has been done on this subject. Despite rich profound results in the framework of Riemannian geometry, the problem in the sub-Riemannian world is more interesting, but far away from being absolutely understood while some conclusive results have been obtained recently. In this report, we will discuss some inequalities of HLS type on I-type groups, the nilpotent part in the Iwasawa decomposition of rank one semisimple Lie groups. For simplicity, we will not give the explicit sharp constants, usually the spectral gap of related operators, for which the interested readers can find in the references.

In the seminal paper [Lie83], Lieb obtained the sharp HLS on  $\mathbb{R}^n$  and  $\mathbb{S}^n$  with all extremizers identified: for singular exponent  $0 < \lambda < n$ ,

$$\iint_{(\mathbb{R}^n)^2} \frac{f(x)f(y)}{|x-y|^\lambda} dx dy \leq C_{n,\lambda}^{\mathbb{R}^n} \|f\|_p \|g\|_p, \quad p = \frac{2n}{2n-\lambda}.$$

A compactness and rearrangement argument play a crucial role, which become a standard method later for related variational problems. A unified competing symmetry method for the existence and extremizers identification was given by

Carlen and Loss [CL90]. They constructed a special strong limit using alternatively the conformal action and the rearrangement to any positive  $L^p$  function, which ingeniously balanced between the “bad” and “good” roles of the symmetry group. Other symmetric rearrangement-free methods can be found in the work of Frank and Lieb [FL10, FL12a]. The first reference used inversion-positivity to get a result for partial exponent  $\lambda \geq n - 2$ , while the second one demonstrated on  $\mathbb{R}^n$  the method used for the Heisenberg group in a recent breakthrough [FL12b], which will also give the extensions in this report. The involves the proof of a quadratic inverse second variation inequality, in the spirit of [Her70, CY87, CY95, BFM13]. The first main theorem is as follows which was announced and stated in [CLZ16a, CLZ16b]:

**Theorem 1** (Sharp HLS). *On a I-type group  $G$  with homogeneous dimension  $Q$ , centre dimension  $d$ , homogeneous norm  $|\cdot|$  and left invariant Haar measure  $du$ , associated to the Lie structure, for any  $2(d-1)_+ < \lambda < Q$  and  $p = 2Q/(2Q - \lambda)$ , we have the following sharp inequality*

$$(1) \quad \iint_{G^2} \frac{f(u)g(v)}{|u^{-1}v|^\lambda} dudv \leq C_{Q,d,\lambda} \|f\|_p \|g\|_p,$$

with “unique” extremizer  $f \approx g \approx J_C^{1/p}$  up to the conformal symmetry group, and  $J_C$  is the Jacobian of associated Cayley transform.

*Remark.* The I-type group is a special class of H-type, requiring additional property of the Lie structure. From the study of the Clifford modulo, it is known that, isomorphically,  $G = \mathbb{K}^n \times \text{Im } \mathbb{K}$  with  $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ , corresponding respectively to the case  $d = 0, 1, 3, 7$ . Especially, the case  $d = 7$  happens only with  $n = 1$ . For notations, we define the element  $u = (z, t)$  with vectors  $z \in \mathbb{K}^n, t \in \text{Im } \mathbb{K}$ , the multiplication  $uu' = (z, t)(z', t') = (z + z', t + t' + 2 \text{Im } z \cdot \bar{z}')$ , Haar measure  $du = dzdt$ , homogeneous dimension  $Q = (d+1)n + 2d$  and norm  $|u| = (|z|^4 + |t|^2)^{1/4}$ . We always use the respective multiplication associated to the division algebra  $\mathbb{K}$  through this report. Besides the left translation from the group multiplication, there are other automorphisms: dilation  $\delta u = (\delta z, \delta^2 t)$  and inversion  $u \mapsto (\frac{-z}{|z|^2 - t}, \frac{-t}{|u|^4})$ , and all of these generate the whole automorphism group  $\text{Aut}(G)$ , also named the conformal transformation group with different language. A typical structure of HLS is the conformal invariance under  $\sigma(f) = f \circ \sigma J_\sigma^{1/p}$  for any  $\sigma \in \text{Aut}(G)$ . There is a natural Cayley transform from the noncompact parameterized group  $G$  and the compact spheres  $\mathbb{S}^{Q-d}$ .  $\mathcal{C}u = (\frac{2z}{1+|z|^2-t}, \frac{1-|z|^2+t}{1+|z|^2-t})$ , with Jacobian  $J_C = 2^{Q-d}((1+|z|^2)^2 + |t|^2)^{-Q/2}$ . This is a generalization of the Stereographic projection between  $\mathbb{R}^n$  and  $\mathbb{S}^{2n+1}$ . By the Cayley transform, there exists an equivalent inequality on the spheres, Corollary 2. Above theorem is also a generalization of former works for  $\lambda = Q - 2$  of Garofalo-Vassilev [GV01], discarding the partial symmetry there, of Ivanov-Minchev-Vassilev [IMV10, IMV12] for quaternionic case, and of the seminal work of Jerison-Lee [JL88] for the Heisenberg case. We remark finally that in the case  $d > 1$ , conclusion holds for the endpoint  $\lambda = 2(d-1)$  and we will fix  $0 < \lambda \in [2(d-1)_+, Q)$  through this report.

**Corollary 2** (Spherical HLS). *On  $\mathbb{S}^{Q-d} \subset \mathbb{K}^{n+1}$ , we have sharp inequality*

$$\iint_{(\mathbb{S}^{Q-d})^2} \frac{F(\zeta)G(\eta)}{|1 - \zeta \cdot \bar{\eta}|^{\lambda/2}} d\zeta d\eta \leq 2^{\lambda(2-d/Q-d/2)} C_{Q,d,\lambda} \|F\|_p \|G\|_p,$$

with all extremizers  $G \approx F(\zeta) \approx |1 - \xi \cdot \bar{\zeta}|^{-(2Q-\lambda)/2}$  with  $\xi \in B_1(\mathbb{K}^{n+1})$ .

*Remark.* Here we use the standard surface Lebesgue measure  $d\zeta$ . The inequality involves a relation between the distances on the compact and noncompact pictures  $2^{1-d/2}|1 - \zeta \cdot \bar{\eta}|^{1/2} = 2^{d/Q-1} J_C(u)^{1/(2Q)} J_C(v)^{1/(2Q)} |u^{-1}v|$ .

Following this sharp HLS, some related sharp inequalities are obtained.

**Corollary 3** (Sharp fractional Sobolev inequality). *We have sharp inequality for functions in the Sobolev space  $H^{s/2}$ ,*

$$\int_G f L_s f du \geq C_{Q,d,\lambda}^{-1} \|f\|_{p'}^2,$$

with  $s + \lambda = Q$  and  $1/p + 1/p' = 1$ , where  $L_s$  is a “fractional subLaplacian” with fundamental solution  $|u|^{-\lambda}$ . Extremizer is “unique”, and equals  $J_C^{1/p'}$ .

*Remark.* The Sobolev space  $H^{s/2}$  and related norm  $\|\cdot\|_{H^{s/2}}$  is induced by the left side functional in above inequality. Similarly, we can define a distribution space  $H^{-s/2}$  and HLS norm  $\|\cdot\|_{H^{-s/2}}$  associated to the HLS functional. In the Euclidean case,  $L_s \approx (-\Delta)^{s/2}$ . Otherwise, we have a spectral characterization of the fractional subLaplacian  $L_s$  in terms of the subLaplacian and the left variant vector field along the central variable, using the spherical Fourier transform, which is another equivalent alternative definition. An analogous fractional conformal subLaplacian  $A_s$  on the spheres will give a spherical version inequality. These operators intertwine with the conformal actions, so we say they are both conformally invariant, and can be considered as the intertwining operators related to the complementary series representations of classical semisimple Lie groups. Note that, the sharp constants in above inequalities are all given by the spectral gaps of associated differential and integral operators, up to a normalization.

**Corollary 4** (Spherical Log-Sobolev inequality). *We have the following sharp inequality for  $0 \leq F \in L^2 \log L(\mathbb{S}^{Q-d})$  normalized by  $\int_{\mathbb{S}^{Q-d}} F^2 = |\mathbb{S}^{Q-d}|$ ,*

$$\iint_{(\mathbb{S}^{Q-d})^2} \frac{|F(\zeta) - F(\eta)|^2}{|1 - \zeta \cdot \bar{\eta}|^{Q/2}} d\zeta d\eta \geq C_{Q,d}^{\log \text{Sob}} \int_{\mathbb{S}^{Q-d}} F^2 \log F^2 d\zeta,$$

with some extremizers  $F(\zeta) \approx |1 - \xi \cdot \bar{\zeta}|^{-Q/2}$ ,  $\xi \in B_1(\mathbb{K}^{n+1})$ , after normalization.

*Remark.* The singular integral in the left side measures the smoothness, and can be characterized by a differential operator obtained by functional differentiation of the fractional conformal subLaplacian  $A_s$  at the endpoint  $s = 0$ , see [Bec97] for classical analogue. For  $d = 0, 1$ , there exists at the other endpoint  $s = n$  or  $\lambda = 0$  the sharp Beckner-Onofri and Log-HLS inequalities [Bec93, BFM13]. For  $d = 3, 7$ , we anticipate the sharp HLS also holds for  $0 < \lambda < 2(d-1)$ , and so is the purported Log-HLS. A group version of the Corollary also holds.

Another main result, obtained in [LZ15], is about the improvements of the sharp inequalities obtained in the first main theorem. We derive a stability for the extremizers of the HLS and fractional Sobolev inequalities, extending [CFW13]. Besides, we compare the dual remainder terms as [DJ14, JH14]. Denote  $M_{\text{HLS}}, M_{\text{FS}}$  the manifolds of all extremizers for the HLS and fractional Sobolev inequalities.

**Theorem 5** (Improvements). *There exists a positive constant  $\alpha = \alpha(Q, d, \lambda)$ , s.t.*

$$C_{Q,d,\lambda} \|f\|_p^2 - \|f\|_{H^{-s/2}}^2 \geq \alpha \inf_{g \in M_{\text{HLS}}} \|f - g\|_p^2,$$

and

$$\|f\|_{H^{s/2}}^2 - C_{Q,d,\lambda}^{-1} \|f\|_{p'}^2 \geq \alpha \inf_{g \in M_{\text{FS}}} \|f - g\|_{H^{s/2}}^2.$$

Besides, concerning the dual remainder terms for  $f \geq 0$ , we have the global estimate

$$\|f\|_{p'}^{2(p'-2)} (\|f\|_{H^{s/2}}^2 - C_{Q,d,\lambda}^{-1} \|f\|_{p'}^2) \geq C_{Q,d,\lambda}^{-2} (C_{Q,d,\lambda} \|f^{p'/p}\|_p^2 - \|f^{p'/p}\|_{H^{-s/2}}^2)$$

and the local estimate

$$\liminf_{f \xrightarrow{H^{s/2}} M_{\text{FS}}^*} \frac{\|f\|_{p'}^{2(p'-2)} (\|f\|_{H^{s/2}}^2 - C_{Q,d,\lambda}^{-1} \|f\|_{p'}^2)}{C_{Q,d,\lambda} \|f^{p'/p}\|_p^2 - \|f^{p'/p}\|_{H^{-s/2}}^2} \geq \frac{Q + 2 + 2 \operatorname{sign} d + s}{Q + 2 + 2 \operatorname{sign} d - s} C_{Q,d,\lambda}^{-2}.$$

*Remark.* Theorem holds both on the groups and spheres. The stability is established from the local stability and a contradiction argument, and therefore we have no explicit constant. A constructive method is still open for global stability. The global dual estimate is not far from the completion of square trick, while the local one is a spectral estimate. They give a lower and upper bound for the sharp constant of the quotient functional in last line. In the case  $d = 1, 2$ , we have similar dual estimate for the Log-HLS and Beckner-Onofri inequalities.

## 2. SKETCH OF THE PROOF

*Proof of Theorem 1.* Following closely Frank-Lieb's argument, we split the proof to several steps.

*Step 1.* We need the existence of extremizers first. A generic method for existence is the concentration compactness technique developed by Lions. We can also follow the compactness+TT\* method as in [FL12b]. In this direction, the positivity of the fractional integral operator for  $\lambda > 2(d-1)_+$  is important. This also restrict the problem to positive functions (and  $F \approx G$ ). For the endpoint of cases with high dimensional centres, we can get the existence from the uniqueness of positive situation, and for uniqueness, we will consider the quadratic extremizer first (assuming  $F = G$ ). See Proposition 9 for the positivity.

*Step 2.* Aiming to the uniqueness, we consider the problem on the sphere for simplicity. By the zero center-mass trick, from Chang-Yang, it suffices to prove that constant is the unique extremizer satisfying the zero center-mass condition.

**Lemma 6.** *For any extremizer  $F$ , there exists a conformal transformation  $\tau \in \operatorname{Aut}(\mathbb{S}^{Q-d})$ , s.t., the new function  $H = F \circ \tau J_\tau^{1/p}$  has zero center-mass, i.e.,*

$$\int_{\mathbb{S}^{Q-d}} H(\zeta) \zeta \, d\zeta = 0.$$

It is interesting that the zero center-mass trick can assimilate such a big conformal symmetry group perfectly. It has also been used recently for the extremal problem of Log-HLS on the Heisenberg group [BFM13]. A constructive proof of this lemma using the group dilations and sphere rotations also helps to see clearly the explicit formulae for extremizers. Be aware of this, we first *claim* that

**Proposition 7.** *Any extremizer with zero center-mass is a constant function.*

*Step 3.* After computation of the second variation of the HLS functional, around the extremizer  $H$  with zero center-mass, along perturbation  $\varphi_j^i = H\zeta_j^i$  with complex coordinates  $\zeta_j^i$ , by summation and symmetry, we can get inequality

$$\iint_{(\mathbb{S}^{Q-d})^2} \frac{H(\zeta)H(\eta)2\operatorname{Re}(\zeta, \eta)_{\mathbb{C}}}{|1 - \zeta \cdot \bar{\eta}|^{\lambda/2}} d\zeta d\eta \leq 2(p-1) \iint_{(\mathbb{S}^{Q-d})^2} \frac{H(\zeta)H(\eta)}{|1 - \zeta \cdot \bar{\eta}|^{\lambda/2}} d\zeta d\eta,$$

where  $(\zeta, \eta)_{\mathbb{C}}$  is the multiplication as complex vectors, whose real part however is equal to that of  $\bar{\zeta} \cdot \eta$  using the standard  $\mathbb{K}$  multiplication.

*Step 4.* We finally prove an inverse of above second variation inequality, which reaches equality only when  $H$  is a constant. This involves meticulous spectral computation. By abuse of notations, we will write the decomposition of  $L^2(\mathbb{S}^{Q-d})$  to a sequence of spherical harmonics  $V_{j,k}$ , with binary subscripts  $\{(j, k)\} \in I$  for all cases with different interpretations, and see references for details in case.

**Proposition 8.** *For  $0 < \lambda \in [2(d-1)_+, Q)$ , the following inequality holds for any  $F \in L^2(\mathbb{S}^{Q-d})$ ,*

$$\iint_{(\mathbb{S}^{Q-d})^2} \frac{F(\zeta)F(\eta)2\operatorname{Re}(\zeta, \eta)_{\mathbb{C}}}{|1 - \zeta \cdot \bar{\eta}|^{\lambda/2}} d\zeta d\eta \geq 2(p-1) \iint_{(\mathbb{S}^{Q-d})^2} \frac{F(\zeta)F(\eta)}{|1 - \zeta \cdot \bar{\eta}|^{\lambda/2}} d\zeta d\eta,$$

*which reaches equality if and only if  $F$  is a constant function when  $\lambda > 2(d-1)_+$ ; When  $\lambda = 2(d-1)$  with  $d > 1$ , equality holds if and only if  $F \in V_{0,0} \oplus V_{j \geq k \geq 2}$ .*

To prove the inverse quadratic inequality, we need to compute the spectrum of the fractional integral operators of two types

$$K_1^\lambda = \frac{1}{|1 - \zeta \cdot \bar{\eta}|^{\lambda/2}} \quad \text{and} \quad K_2^\lambda = \frac{|\zeta \cdot \bar{\eta}|^2}{|1 - \zeta \cdot \bar{\eta}|^{\lambda/2}},$$

as

$$\frac{2\operatorname{Re}(\zeta, \eta)_{\mathbb{C}}}{|1 - \zeta \cdot \bar{\eta}|^{\lambda/2}} = \frac{2\operatorname{Re}\bar{\zeta} \cdot \eta}{|1 - \zeta \cdot \bar{\eta}|^{\lambda/2}} = \frac{1}{|1 - \zeta \cdot \bar{\eta}|^{\lambda/2}} + \frac{|\zeta \cdot \bar{\eta}|^2}{|1 - \zeta \cdot \bar{\eta}|^{\lambda/2}} - \frac{1}{|1 - \zeta \cdot \bar{\eta}|^{\lambda/2-2}}.$$

We list the result without proof, which involves a Funk-Hecke type formula.

**Proposition 9 (Eigenvalues).** *The eigenvalues of the two kernels w.r.t. the decomposition of spherical harmonics are respectively*

$$E_{j,k}^1 = \frac{2\pi^{\frac{Q-d}{2}+1}\Gamma((Q-\lambda)/2)}{\Gamma(\lambda/4)\Gamma(\lambda/4-d/2+1)} \frac{\Gamma(j+\lambda/4)}{\Gamma(j+(2Q-\lambda)/4)} \frac{\Gamma(k+\lambda/4+(1-d)/2)}{\Gamma(k+(2Q-\lambda)/4+(1-d)/2)},$$

and

$$E_{j,k}^2 = E_{j,k}^1 \left[ 1 - (\lambda/4 - (1+d)/2)(c-a-b) \left( \frac{1}{(a-1)(c-a)} + \frac{1}{(b-1)(c-b)} - (\lambda/4 - (1+d)/2) \frac{c-a-b+1}{(a-1)(b-1)(c-a)(c-b)} \right) \right]$$

with

$$(a, b, c) := (j + \lambda/4, k + \lambda/4 + (1-d)/2, j + k + Q/2 + (1-d)/2).$$

*Step 5. Conclusion.* For positive case  $\lambda > 2(d-1)_+$ , from Step 3-4, we know any extremizer with zero center-mass has to be a constant, then from Step 2, we know any extremizer can be written as  $J_\tau^{1/p}$  for some  $\tau$ , which indeed is, for any  $\tau$  from the conformal invariance. For semipositive case  $\lambda = 2(d-1)$  with  $d > 1$ , we

can borrow the Euler-Lagrange equation to prove the claim, concerning the last sentence in Proposition 8. Explicit formulae for the extremizers and Jacobian of the conformal transformations can be calculated both on the groups and spheres. So far, we have proved Theorem 1 and Corollary 2.  $\square$

*Proof of Corollary 3.* We consider the problem on the sphere and assume that  $A_s^{-1} = |1 - \zeta \cdot \bar{\eta}|^{-\lambda/2}$ . Using  $L^2$  inner product  $(\cdot, \cdot)$  and from Cauchy-Schwarz inequality, we have a dual argument,

$$\begin{aligned} \|F\|_{p'}^2 &= \sup_{\|G\|_p=1} (F, G)^2 = \sup_{\|G\|_p=1} (A_s^{1/2}F, A_s^{-1/2}G)^2 \\ &\leq \sup_{\|G\|_p=1} (F, A_s F) (G, A_s^{-1}G) = C_{Q,d,\lambda} (F, A_s F) \end{aligned}$$

which reaches equality if and only if  $G$  is an extremizer for the spherical HLS and  $F = G^{p/p'}$ , that is  $F = J_\tau^{1/p'}$  for any conformal transformation  $\tau$  on the spheres.  $\square$

*Proof of Corollary 4.* We use the kind of functional differential argument, just as for the classical case. Note that

$$\iint_{(\mathbb{S}^{Q-d})^2} \frac{F^2(\zeta) + F^2(\eta)}{|1 - \zeta \cdot \bar{\eta}|^{\lambda/2}} d\zeta d\eta = 2 E_{0,0}^1 |\mathbb{S}^{Q-d}| = 2 C_{Q,d,\lambda}^{\text{SHLS}} |\mathbb{S}^{Q-d}|^{2/p},$$

and subtract double the sharp spherical HLS, we get

$$\begin{aligned} \iint_{(\mathbb{S}^{Q-d})^2} \frac{|F(\zeta) - F(\eta)|^2}{|1 - \zeta \cdot \bar{\eta}|^{\lambda/2}} d\zeta d\eta &\geq 2 C_{Q,d,\lambda}^{\text{SHLS}} (|\mathbb{S}^{Q-d}|^{2/p} - \|F\|_p^2) \\ &\xrightarrow{\lambda \rightarrow Q} C_{Q,d}^{\log \text{Sob}} \int_{\mathbb{S}^{Q-d}} F^2 \log F^2 d\zeta. \end{aligned}$$

It is done after checking the limitations. Actually, we can also differentiate the sharp fractional Sobolev inequality at  $s = 0$  to get this inequality.  $\square$

*Proof of Theorem 5.* We consider the problems on the sphere.

*Stability.* We need the following two lemmas, a local stability result and a compactness argument. Denote  $E_1 = E_{1,1}^1$  for  $d = 0$ , and  $E_1 = E_{1,0}^1$  for  $d > 0$  and similarly define  $E_2$ , using the eigenvalues in Proposition 9. Then

$$E_1/E_2 = \frac{Q + s + 2 + 2 \text{sign } d}{Q - s + 2 + 2 \text{sign } d} \quad \text{and} \quad 1 - E_2/E_1 = \frac{2s}{Q + s + 2 + 2 \text{sign } d}.$$

**Lemma 10** (Local stability). *Denote  $d(F, G)$  the corresponding distance norms. We have the following stability estimates in the local regime:*

- (1) For  $\|F\|_{H^{s/2}} > d(F, M_{\text{FS}}) \rightarrow 0$ ,
 
$$\begin{aligned} d^2(F, M_{\text{FS}}) &\geq \|F\|_{H^{s/2}}^2 - C_{Q,d,\lambda}^{-1} \|F\|_{p'}^2 \\ &\geq (1 - E_2/E_1) d^2(F, M_{\text{FS}}) + o(d^2(F, M_{\text{FS}})). \end{aligned}$$
- (2) There exist two positive constants  $\alpha_0$  and  $\alpha_1$ , only depending on  $Q, d$  and  $\lambda$ , s.t., for  $F \in L^p$  with  $d(F, M_{\text{HLS}}) \rightarrow 0$  and  $F \not\rightarrow 0$ ,
 
$$\begin{aligned} \|F\|_p (\alpha_1 d(F, M_{\text{HLS}}) + o(d(F, M_{\text{HLS}}))) &\geq C_{Q,d,\lambda} \|F\|_p^2 - \|F\|_{H^{-s/2}}^2 \\ &\geq \alpha_0 d^2(F, M_{\text{HLS}}) + o(d^2(F, M_{\text{HLS}})) \end{aligned}$$

*Remark.* The first estimate is not far from orthogonal decomposition, the Taylor expansion and spectral estimates. The second one is more complicated as the difference functional is not in  $C^2$ , because  $p < 2$ . This problem has been dealt in a former work of Christ for Hausdorff-Young inequality. We can handle similarly or borrow the second variation lemma there directly.

**Lemma 11** (Compactness). *Let  $(F_j)$  be an extremizing sequence of the HLS or fractional Sobolev inequalities, i.e.*

$$\|F_j\|_{H^{-s/2}}^2 / \|F_j\|_p^2 \xrightarrow{j} C_{Q,d,\lambda}, \quad \text{or} \quad \|F_j\|_{H^{s/2}}^2 / \|F_j\|_{p'}^2 \xrightarrow{j} C_{Q,d,\lambda}^{-1},$$

then

$$d(F_j, M_{\text{HLS}}) / \|F_j\|_p \xrightarrow{j} 0, \quad \text{or} \quad d(F_j, M_{\text{FS}}) / \|F_j\|_{H^{s/2}} \xrightarrow{j} 0,$$

*Remark.* This lemma is about the precompactness of the extremizing sequence, which is not trivial, and then gives the existence of extremizer, a strong limit of some extremizing sequence.

*Conclusion.* By contradiction, if the global positive constant  $\alpha$  in the right side in Theorem 5, does *not* exist, then we have a sequence  $(F_j)$ , s.t., the quotient of the remainder term and the distance square of  $(F_j)$  will goes to zero. This means that it is an extremizing sequence as  $d(F, M)$  is less than the norm of  $F$ , which means, by the compactness lemma, it is in the neighborhood of the extremizing manifold. This amounts to the local stability estimate, which will give a positive lower bound of the remainder terms, immediately a contradiction.

*Dual estimates.* The global estimate comes from the trick of completion of squares,

$$\| \|F\|_{p'}^{p'-2} A_s^{1/2} F - C_{Q,d,\lambda}^{-1} A_s^{-1/2} (F^{p'/p}) \|_2^2 \geq 0,$$

while the local estimate comes from Taylor expansion and spectral estimates. Assume  $F = 1 + \varphi$  with  $\varphi \perp T_1 M = H_0 \oplus H_1$  where  $H_k$  is the classical spherical harmonics on the real sphere. Then the quotient functional can be estimated by linearization using  $\varphi$ . Take the spherical harmonics decomposition  $\varphi = \sum_I \varphi_{j,k}$ , then with some constant sequences  $(c_{j,k})$ ,

$$\begin{aligned} & \liminf_{\varphi \rightarrow 0, \varphi \perp T_1 M} \frac{\|F\|_{p'}^{2(p'-2)} (\|F\|_{H^{s/2}}^2 - C_{Q,d,\lambda}^{-1} \|F\|_{p'}^2)}{C_{Q,d,\lambda} \|F^{p'/p}\|_p^2 - \|F^{p'/p}\|_{H^{-s/2}}^2} \\ &= \liminf E_1 \frac{\sum_{\varphi \perp T_1 M} c_{j,k} \|\varphi_{j,k}\|_2^2}{\sum_{\varphi \perp T_1 M} E_{j,k} c_{j,k} \|\varphi_{j,k}\|_2^2} = E_1 / E_2. \end{aligned}$$

It is done and in cases  $d = 0, 1$ , this can be done for the dual Log-HLS and Beckner-Onofri inequalities.  $\square$

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