Séminaire Laurent Schwartz

EDP et applications

Année 2014-2015

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Séminaire Laurent Schwartz — EDP et applications (2014-2015), Exposé n° XVIII, 17 p.

<http://slsedp.cedram.org/item?id=SLSEDP_2014-2015____A18_0>

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Séminaire Laurent-Schwartz — EDP et applications
Centre de mathématiques Laurent Schwartz, 2014-2015
Exposé n^{o} XVIII, 1-17

The parabolic-parabolic Keller-Segel equation

Kleber Carrapatoso*†

Abstract

I present in this note recent results on the uniqueness and stability for the parabolic-parabolic Keller-Segel equation on the plane, obtained in collaboration with S. Mischler in [11].

1 Introduction

1.1 The model

The Keller-Segel (KS) system (or the Patlak-Keller-Segel system) is a model in chemotaxis that describes the collective motion of cells that are attracted by a chemical substance that they are able to emit ([27, 20]). We consider in this note the parabolic-parabolic KS equation in the plane, which takes the form

$$\begin{cases} \partial_t f = \Delta f - \nabla \cdot (f \nabla u) & \text{on} \quad (0,\infty) \times \mathsf{R}^2, \\ \epsilon \partial_t u = \Delta u + f - \alpha u & \text{on} \quad (0,\infty) \times \mathsf{R}^2, \\ f|_{t=0} = f_0, \ u|_{t=0} = u_0 & \qquad \text{in} \quad \mathsf{R}^2, \end{cases} \tag{1.1}$$

and we shall present recent results on the uniqueness and stability of (1.1), obtained in collaboration with S. Mischler in [11]. The function $f = f(t,x) \geq 0$ represents the mass density of cells while $u = u(t,x) \geq 0$ stands for the chemo-attractant concentration, $t \in \mathsf{R}^+$ is the time variable, $x \in \mathsf{R}^2$ is the space variable, and $\epsilon > 0$, $\alpha \geq 0$ are constants. For a biological motivation and mathematical introduction, we refer to [7] and the references therein.

We also mention that when $\epsilon = 0$ we obtain the parabolic-elliptic KS system, and we refer the reader to [5, 9, 15] for more details.

1.2 Basic properties

Let us present some basic properties of the parabolic-parabolic KS equation. At least formally, solutions to (1.1) satisfy two fundamental identities. First of all,

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we have the conservation of mass

$$\int_{\mathbf{R}^2} f(t,x)\,dx = \int_{\mathbf{R}^2} f_0(x)\,dx =: M, \quad \forall\, t \geq 0. \tag{1.2}$$

Moreover, we have the free energy-dissipation of the free energy identity

$$\mathcal{F}(t) + \int_0^t \mathcal{D}_{\mathcal{F}}(s)\,ds = \mathcal{F}_0, \tag{1.3}$$

where the free energy $\mathcal{F}(t) = \mathcal{F}(f(t),u(t))$, $\mathcal{F}_0 = \mathcal{F}(f_0,u_0)$, is defined by

$$\mathcal{F} = \mathcal{F}(f,u) := \int_{\mathbf{R}^2} f \log f\,dx - \int_{\mathbf{R}^2} f u\,dx + \frac{1}{2}\int_{\mathbf{R}^2} |\nabla u|^2\,dx + \frac{\alpha}{2}\int_{\mathbf{R}^2} u^2 dx,$$

and the dissipation of the free energy $\mathcal{D}_{\mathcal{F}}(s) = \mathcal{D}_{\mathcal{F}}(f(s),u(s))$ by

$$\mathcal{D}_{\mathcal{F}} = \mathcal{D}_{\mathcal{F}}(f,u) := \int_{\mathbf{R}^2} f\,|\nabla(\log f) - \nabla u|^2\,dx + \frac{1}{\epsilon}\int_{\mathbf{R}^2} |\Delta u + f - \alpha\, u|^2\,dx.$$

We shall always assume that the initial data (f_0,u_0) satisfy

$$\begin{cases} f_0\,(1+\log\langle x\rangle^2) \in L^1(\mathbf{R}^2) \quad \text{and} \quad f_0 \log f_0 \in L^1(\mathbf{R}^2); \\ u_0 \in H^1(\mathbf{R}^2) \text{ if } \alpha > 0 \quad \text{or} \quad u_0 \in L^1(\mathbf{R}^2) \cap \dot{H}^1(\mathbf{R}^2) \text{ if } \alpha = 0; \\ f_0\,u_0 \in L^1(\mathbf{R}^2), \end{cases} \tag{1.4}$$

where here and below we define the weight function $\langle x\rangle := (1+|x|^2)^{1/2}$, and we also make the restriction to the subcritical mass case

$$M = \int_{\mathbf{R}^2} f_0(x)\,dx \in (0, 8\pi), \tag{1.5}$$

since under this hypothesis there is a global existence theory (see [7] or Theorem 2.2 below). It is worth mentioning other available existence results: we refer to [19, 25, 24] for blow-up solutions when $M > 8\pi$; and also to [4, 13] for a global existence theory in the possible supercritical case $M > 8\pi$ under the condition that $\epsilon > 0$ is large enough, which corresponds to a small nonlinearity in (1.1).

1.3 Structure of the note

In section 2 we present a result on regularisation and uniqueness of weak solutions to (1.1) in Theorem A. Then, in section 3, we present a result concerning the nonlinear stability of the self-similar profile in Theorem B.

2 Regularisation and uniqueness

Let us now define the notion of weak solutions we shall consider.

Definition 2.1 ([7]). For any initial datum (f_0, u_0) satisfying (1.4)-(1.5), we say that the couple (f, u) of nonnegative functions satisfying

$$
\begin{aligned}
& f \in L^\infty(0,T;L^1(\mathbf{R}^2)) \cap C([0,T);\mathcal{D}'(\mathbf{R}^2)), \quad \forall T \in (0,\infty), \\
& \begin{cases} u \in L^\infty(0,T;H^1(\mathbf{R}^2)) & \text{if } \alpha > 0; \\ u \in L^\infty(0,T;L^1(\mathbf{R}^2) \cap \dot H^1(\mathbf{R}^2)) & \text{if } \alpha = 0; \end{cases} \\
& fu \in L^\infty(0,T;L^1(\mathbf{R}^2))
\end{aligned}
$$

is a global in time weak solution to the Keller-Segel equation (1.1) associated to the initial condition (f_0, u_0) whenever (f, u) satisfies the mass conservation (1.2), the bound

$$\sup_{[0,T]} \mathcal{F}(t) + \sup_{[0,T]} \int_{\mathbf{R}^2} f \, \log\langle x\rangle^2 \, dx + \int_0^T \mathcal{D}_{\mathcal{F}}(t)\, dt \le C_T, \tag{2.1}$$

as well as the Keller-Segel system (1.1) in the distributional sense.

Let us briefly explain, following [7], how to obtain the basic estimates which lead to the notion of weak solution as presented in Definition 2.1. We first observe that the following space logarithmic moment control holds true

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbf{R}^2} f\,(-\log H)\,dx &= \int_{\mathbf{R}^2} f\,\nabla(\log f - u)\cdot\nabla(\log H)\,dx \\
&\le \frac{1}{2}\mathcal{D}_{\mathcal{F}}(f,u) + \frac{1}{2}\int_{\mathbf{R}^2} f\,|\nabla \log H|^2\,dx,
\end{aligned}
$$

where we have defined $H(x) := \frac{1}{\pi}\frac{1}{\langle x\rangle^4}$ whence $|\nabla \log H(x)| \le 4$, which together with (1.3) imply that the modified free energy functional

$$\mathcal{F}_H = \mathcal{F}(f,u) - \int_{\mathbf{R}^2} f\,\log H$$

satisfies

$$\frac{d}{dt}\mathcal{F}_H(t) + \frac{1}{2}\mathcal{D}_{\mathcal{F}}(t) \le M. \tag{2.2}$$

On the one hand, introducing the Laplace kernel $\kappa_0(z) := -\frac{1}{2\pi}\log|z|$ and the Bessel kernel $\kappa_\alpha(z) := \frac{1}{4\pi}\int_0^\infty t^{-1}\exp(-|z|^2/(4t) - \alpha t)\,dt$ for $\alpha > 0$, so that $\overline{u}_\alpha := \kappa_\alpha * f$ is a solution to the equation $(-\Delta+\alpha)\overline{u}_\alpha = f$ in $\mathbf{R}^2$, and introducing as well the chemical energy and the modified entropy

$$F_\alpha(f,u) := \frac{1}{2}\int|\nabla u|^2 + \frac{\alpha}{2}\int u^2 - \int f\,u, \quad \mathcal{H}_H(f) := \int f\log(f/H),$$

one can easily show (see e.g. [7, Lemma 2.2]) that

$$\mathcal{F}_H(f,u) = \mathcal{H}_H(f) + F_\alpha(f,\overline{u}_\alpha) + \frac{1}{2}\int|\nabla(u-\overline{u}_\alpha)|^2 + \frac{\alpha}{2}\int(u-\overline{u}_\alpha)^2 \tag{2.3}$$

and

$$F_\alpha(f,\overline{u}_\alpha) = -\frac{1}{2}\iint_{\mathbf{R}^2\times\mathbf{R}^2} f(x)\, f(y)\, \kappa_\alpha(x-y)\, dxdy. \tag{2.4}$$

On the other hand, we know from the classical logarithmic Hardy-Littlewood Sobolev inequality (see e.g. [1, 10]) or its generalisation for the Bessel kernel (see [7, Lemma 4.2]) that, for any $f \geq 0$,

$$\begin{aligned} \int_{\mathbf{R}^2} f(x)\log f(x)\, dx \quad &- \quad \frac{4\pi}{M}\iint_{\mathbf{R}^2\times\mathbf{R}^2} f(x)\, f(y)\, \kappa_\alpha(x-y)\, dxdy \\ &- \quad \int_{\mathbf{R}^2} f(x)\log H(x)\, dx \geq -C_1(M), \end{aligned} \tag{2.5}$$

where here and below $C_i(M)$ denotes a positive constant which only depends on the mass M. Then from (2.3), (2.4) and (2.5) together with the very classical functional inequality (see e.g. [7, Lemma 2.4])

$$\mathcal{H}^+(f) := \int f(\log f)_+ \leq \mathcal{H}_H(f) - \frac{1}{4}\int f\log\langle x\rangle^2 + C_2(M),$$

one immediately obtains, for $M < 8\pi$,

$$\begin{aligned} \mathcal{F}_H(f,u) \geq (1-\frac{M}{8\pi})\,\mathcal{H}_H(f) \\ &+ \frac{M}{8\pi}\Big(\mathcal{H}_H(f) - \frac{4\pi}{M}\iint_{\mathbf{R}^2\times\mathbf{R}^2} f(x)\, f(y)\, \kappa_\alpha(x-y)\, dxdy\Big) \\ &\geq C_3(M)\,\mathcal{H}_+(f) + C_4(M)\int f\log\langle x\rangle^2 - C_5(M). \end{aligned}$$

One concludes that under the assumption (1.4) on the initial datum, the identity (1.2) and the inequality (2.2) provide a convenient family of a priori estimates in order to define weak solutions, namely

$$\begin{aligned} C_3(M)\,\mathcal{H}^+(f(t)) + C_4(M)\int f(t)\log\langle x\rangle^2 + \frac{1}{2}\int_0^t \mathcal{D}_{\mathcal{F}}(f(s),u(s))\, ds \\ \leq \mathcal{F}_H(0) + C_5(M) + M\,t, \end{aligned} \tag{2.6}$$

It is worth emphasising that in order to get the bounds announced in Definition 2.1 in the case $\alpha > 0$ one may use the inequality (see [7, Eq. (3.5)])

$$\mathcal{F}_H \geq C_6(M)\int |\nabla u|^2 + C_7(M)\alpha\int u^2 + C_8(M)\int f\,u - C_9(M)(1+1/\alpha).$$

This framework is well adapted for a global existence theory in the subcritical mass case $M \in (0, 8\pi)$.

Theorem 2.2 ([7, Theorem 1]). *For any initial datum (f_0, u_0) satisfying* (1.4)-(1.5)*, there exists at least one global weak solution in the sense of Definition 2.1 to the Keller-Segel equation* (1.1).

The first main result that we present in this note establishes uniqueness of weak solutions in the same framework as Theorem 2.2.

Theorem A. *For any initial datum* (f_0, u_0) *satisfying* (1.4)-(1.5)*, there exists at most one weak solution in the sense of Definition 2.1 to the Keller-Segel equation* (1.1)*. This one is furthermore a classical solution in the sense that*

$$f, u \in C_b^2((0,\infty) \times \mathbf{R}^2) \tag{2.7}$$

and satisfies the accurate small time estimate

$$\forall\, q \in [4/3, 2), \qquad t^{1-1/q}\, \|f(t)\|_{L^q} \to 0 \quad as \quad t \to 0. \tag{2.8}$$

Finally, the free energy-dissipation of the free energy identity (1.3) *holds.*

The result in Theorem A improves the result of [12] where uniqueness is proven in the class $f \in C([0,T];\, L^1_2(\mathbf{R}^2)) \cap L^\infty((0,T) \times \mathbf{R}^2)$ and under the additional assumption $f_0 \in L^\infty(\mathbf{R}^2)$. We also mention the works [16, 4, 13] where the well-posedness is proved in some particular regimes. Our proof follows a strategy introduced in [17] for the 2D viscous vortex model and generalises a similar result obtained in [15] for the parabolic-elliptic system $\epsilon = 0$.

Overview of the proof

The proof is based on intermediate regularity a posteriori estimates that enable us to use a DiPerna-Lions [14] renormalisation process, which in turn makes possible to first obtain the estimate (2.7) and then to get the optimal regularity of solutions for small time (2.8). After that, we follow the uniqueness argument introduced by Ben-Artzi for the 2D viscous vortex model [2, 6].

2.1 A posteriori estimates

Let (f, u) be any weak solution in the sense of Definition 2.1. We shall explain how to get estimate (2.7) as a consequence of intermediate a posteriori regularity bounds.

We define the Fisher information of f by

$$I(f) := \int_{\mathbf{R}^2} \frac{|\nabla f|^2}{f}\, dx.$$

Thanks to the integral in time bound of the dissipation of the free-energy $\mathcal{D}_{\mathcal{F}}(f)$ and the L^∞-bound on (the positive part of) the entropy $\mathcal{H}^+(f)$ (see (2.1) and (2.6)), we first obtain that

$$I(f(t)) \in L^1(0,T), \quad \forall\, T > 0.$$

As a consequence of previous bound, together with the conservation of mass and some classical inequalities (Hölder and Sobolev in dimension $d = 2$), we obtain some integrability and regularity bounds for the weak solution (f,u), more precisely

$$\begin{aligned} &f \in L^{p/(p-1)}(0,T;L^p(\mathbf{R}^2)), \quad \forall\, p \in (1,\infty),\\ &\nabla f \in L^{2p/(3p-2)}(0,T;L^p(\mathbf{R}^2)), \quad \forall\, p \in [1,2),\\ &\Delta u \in L^2(0,T;L^2(\mathbf{R}^2)). \end{aligned} \tag{2.9}$$

In particular we have already obtained that any weak solution (f,u) satisfies $f \in L^2(0,T;L^2(\mathbf{R}^2))$ and $\nabla u \in L^2(0,T;W^{1,2}(\mathbf{R}^2))$. Therefore we can apply the renormalisation argument of DiPerna-Lions [14], because the latter bounds provides that the commutation Lemma [14, Lemma II.1 and Remark 4] holds true. We hence obtain the following result.

Lemma 2.3. *Any weak solution (f,u) satisfies*

$$\begin{aligned} &\int_{\mathbf{R}^2} \beta(f(t_1))\,dx + \int_{t_0}^{t_1}\int_{\mathbf{R}^2} \beta''(f(s))\,|\nabla f(s)|^2\,dx\,ds\\ &\quad\le \int_{\mathbf{R}^2} \beta(f(t_0))\,dx + \int_{t_0}^{t_1}\int_{\mathbf{R}^2} \{\beta(f(s)) - f_s\beta'(f(s))\}\Delta u(s)\,dx\,ds, \end{aligned}$$

for any times $0 \le t_0 \le t_1 < \infty$ and any renormalising function $\beta : \mathbf{R} \to \mathbf{R}$ which is convex, piecewise of class C^1 and such that

$$|\beta(\xi)| \le C\,(1+\xi\,(\log\xi)_+), \quad |\beta(\xi) - \xi\beta'(\xi)| \le C\,\xi \quad \forall\,\xi \in \mathbf{R}.$$

Applying Lemma 2.3 twice for well-chosen (sequences of) renormalising functions β, we are able then to improve the regularity of f obtaining some uniform in time bounds in the following way:

Lemma 2.4. *For any weak solution (f,u), any $p \ge 2$ and any $t_0 \in [0,T)$ such that $f(t_0) \in L^p(\mathbf{R}^2)$, there exists a constant $C := C(M,\mathcal{H}_0,\mathcal{F}_0,T,p,\|f(t_0)\|_{L^p})$ such that, for all $t_0 < t_1 \le T$, there holds*

$$\|f(t_1)\|_{L^p}^p + \frac{1}{2}\int_{t_0}^{t_1} \|\nabla_x f^{p/2}(t)\|_{L^2}^2\,dt \le C.$$

We are able now to prove estimate (2.7). Gathering the regularity estimates (2.9) together with Lemma 2.4 and using the maximal regularity of parabolic equations in L^p-spaces, a bootstrap argument gives us:

Lemma 2.5. *Any weak solution (f,u) satisfies*

$$\partial_t f,\ \partial_x f,\ \partial^2_{x_i x_j} f,\ \partial_t u,\ \partial_x u,\ \partial^2_{x_i x_j} u \in C_b((0,T]\times\mathbf{R}^2),\ \forall\, T>0,$$

so that it is a "classical solution" for positive time.

An easy consequence of last result is that the free energy-dissipation of the free energy identity (1.3) also holds, which can be obtaining by proving by standard techniques that the free energy $\mathcal{F}$ is lower semi-continuous.

2.2 Uniqueness

We now turn to the proof of the uniqueness part of Theorem A. We consider any weak solution (f, u). We shall first explain how to get the optimal regularity estimate (2.8) and finally, from this estimate, how to obtain the uniqueness following the argument of [2].

First of all, as a consequence of Lemma 2.3, we already obtain the following small time estimate:

Lemma 2.6. *Any weak solution (f, u) to the Keller-Segel equation satisfies that for any $p \in (1, \infty)$, $T \in (0, \infty)$ there exists a constant $K = K(f_0, p, T)$ such that*

$$t^{p-1} \|f(t)\|^p_{L^p} \le K \qquad \forall\, t \in (0, T).$$

Thanks to an interpolation argument with the uniform in time bound of $\mathcal{H}^+(f)$, we can crucially improve the preceding estimate and obtain (2.8).

Lemma 2.7. *For any $p \in [2, \infty)$, any weak solution (f, u) to the Keller-Segel equation satisfies:*

$$t^{(p-1)/2p} \|f(t)\|_{L^{2p/(p+1)}} \to 0 \quad \text{as } t \to 0.$$

We are now able to prove the uniqueness of solutions. Let (f_1, u_1) and (f_2, u_2) be two weak solutions to the Keller-Segel equation (1.1). Assuming that $f_1(0) = f_2(0)$ and $u_1(0) = u_2(0) = u_0$, the difference $F := f_2 - f_1$ satisfies in the mild form

$$\begin{aligned} F(t) = &- \int_0^t \nabla e^{(t-s)\Delta} \left\{ F(s) \left[e^{-\frac{\alpha}{\epsilon} s} e^{\frac{s}{\epsilon}\Delta} (\nabla u_0) \right] \right\} ds \\ &- \int_0^t \nabla e^{(t-s)\Delta} \left\{ F(s) \left[\frac{1}{\epsilon} \int_0^s e^{-\frac{\alpha}{\epsilon}(s-\sigma)} \nabla e^{\frac{(s-\sigma)}{\epsilon}\Delta} f_2(\sigma)\, d\sigma \right] \right\} ds \\ &- \int_0^t \nabla e^{(t-s)\Delta} \left\{ f_1(s) \left[\frac{1}{\epsilon} \int_0^s e^{-\frac{\alpha}{\epsilon}(s-\sigma)} \nabla e^{\frac{(s-\sigma)}{\epsilon}\Delta} F(\sigma)\, d\sigma \right] \right\} ds, \end{aligned}$$

where $e^{t\Delta}$ denotes the heat semigroup. We define then

$$\Delta(t) := \sup_{0<s\le t} s^{1/4} \|F(s)\|_{L^{4/3}}.$$

and, for some fixed $p > 2$,

$$Z^i_p(t) := \sup_{0<s\le t} s^{\frac{1}{2} - \frac{1}{2p}} \|f_i(s)\|_{L^{\frac{2p}{p+1}}}.$$

Thanks to Lemma 2.7 we can obtain that

$$\Delta(t) \le C \big\{ \alpha(t) + Z^1_p(t) + Z^2_p(t) \big\}\, \Delta(t),$$

for some constructive function $\alpha(t) \to 0$ as $t \to 0$. This implies $\Delta(t) \equiv 0$ on $[0, T)$ if $T > 0$ is small enough, and then we repeat the argument for later times to conclude to the uniqueness.

3 Nonlinear stability of self-similar profile

We now restrict to the case $\alpha = 0$ and we focus on the long time behaviour of solutions.

We are interested in self-similar solutions to the parabolic-parabolic KS equation (1.1), which means solutions that can be written as

$$f(t,x) = \frac{1}{t} G\left(\frac{x}{\sqrt{t}}\right), \quad u(t,x) = V\left(\frac{x}{\sqrt{t}}\right),$$

with

$$\int_{\mathbf{R}^2} f(t,x)\, dx = \int_{\mathbf{R}^2} G(y)\, dy = M \in (0, 8\pi).$$

Such a couple of functions (f,u) is a solution to (1.1) if and only if the associated self-similar profile (G, V) satisfies the elliptic system

$$\begin{aligned} \Delta G - \nabla(G \nabla V - \frac{1}{2} x G) = 0 \quad \text{in} \quad \mathbf{R}^2, \\ \Delta V + \frac{\epsilon}{2} x \cdot \nabla V + G = 0 \quad \text{in} \quad \mathbf{R}^2. \end{aligned} \tag{3.1}$$

It is known that, for any $\epsilon \in (0, 1/2)$ and any $M \in (0, 8\pi)$, there exists a unique radially symmetric smooth solution (G, V) to (3.1) such that the mass of G equals M (see [26, 3, 13]).

For this purpose, it is convenient to work in self-similar variables. We hence introduce the rescaled functions g and v defined by

$$\begin{aligned} f(t,x) &:= R(t)^{-2} g(\log R(t), R(t)^{-1} x), \\ u(t,x) &:= v(\log R(t), R(t)^{-1} x), \end{aligned}$$

with $R(t) := (1+t)^{1/2}$. For these new unknowns, the rescaled parabolic-parabolic Keller-Segel system reads

$$\begin{cases} \partial_t g = \Delta g + \nabla(\frac{1}{2} x\, g - g \nabla v) & \text{on} \quad (0,\infty) \times \mathbf{R}^2, \\ \epsilon \partial_t v = \Delta v + g + \frac{\epsilon}{2} x \cdot \nabla v & \text{on} \quad (0,\infty) \times \mathbf{R}^2, \end{cases} \tag{3.2}$$

and therefore the solution (G, V) to (3.1) corresponds to a stationary solution to (3.2).

Our second main result concerns the exponential nonlinear stability of the self-similar profile for any given mass $M \in (0, 8\pi)$ under the strong restriction of radial symmetry and closeness to the parabolic-elliptic regime, i.e. $\epsilon > 0$ small. We define the norm

$$|||(g,v)||| := \|\langle x\rangle^k g\|_{L^2} + \|\langle x\rangle^k \nabla g\|_{L^2} + \|v\|_{H^2}, \quad k > 7.$$

Theorem B. *For any given mass $M \in (0, 8\pi)$, there exist $\epsilon^* > 0$ and $\delta^* > 0$ such that for any $\epsilon \in (0, \epsilon^*)$ and any radially symmetric initial datum (g_0, v_0) satisfying*

$$|||(g_0, v_0) - (G\ , V\ )||| \le \delta^*, \quad \int_{\mathbf{R}^2} g_0 \, dx = \int_{\mathbf{R}^2} G\ \ dx = M,$$

the associated solution (g, v) to (3.2) satisfies

$$|||(g(t), v(t)) - (G\ , V\ )||| \le C_a \, e^{at} \quad \forall \, a \in (-1/3, 0), \ \forall \, t \ge 0,$$

for some constant $C_a = C_a(g_0, v_0)$.

Remark 3.1. Coming back to the original unknowns (f, u), this theorem asserts that

$$f(t, x) \sim \frac{1}{t} \, G \, \left(\frac{x}{\sqrt{t}} \right), \quad u(t, x) \sim V \, \left(\frac{x}{\sqrt{t}} \right), \quad \text{as} \quad t \to \infty.$$

That result extends to the parabolic-parabolic Keller-Segel equation similar results known on the parabolic-elliptic Keller-Segel equation $\epsilon = 0$, see [15]. To our knowledge, Theorem B is the first exponential stability result for the system (1.1) even under the two strong restrictions of radial symmetry and quasi parabolic-elliptic regime (we mean $\epsilon > 0$ small). We also refer to the recent work [13, Section 4] where some results of convergence (without rate) of some solutions to the associated self-similar profile are established.

Overview of the proof

The first part of the proof is the study of spectral and semigroup estimates for the linearised operator associated to (3.2). More precisely, we shall prove that this semigroup is exponentially stable in some weighted $H^1 \times H^2$ space in the quasi-parabolic regime $\epsilon > 0$ small enough, by taking advantage of the exponential stability of the linearised semigroup in the parabolic-elliptic case $(\epsilon = 0)$. Finally, we come back to the nonlinear equation (3.2) and prove that it inherits the exponential stability of the linearised equation if we are close enough to the equilibrium $(G\ , V\ )$.

Notations

For a closed linear operator Λ on a Banach space X, we denote by $D(\Lambda)$ its domain, by $S_\Lambda(t) = e^{t\Lambda}$ the semigroup of operators generated by Λ, by $\sigma(\Lambda)$ its spectrum, by $\sigma_d(\Lambda)$ its discrete spectrum, and by $\mathcal{R}_\Lambda(z)$ the resolvent that is defined for $z \in \mathbf{C}$ in the resolvent set $\rho(\Lambda)$. Moreover, for two Banach spaces X, Y we denote by $\mathscr{B}(X, Y)$ the space of bounded linear operators from X to Y and by $\| \cdot \|_{X \to Y}$ its norm, with the usual shorthand $\mathscr{B}(X) = \mathscr{B}(X, X)$. We also define the subset $\Delta_a \subset \mathbf{C}$ for any $a \in \mathbf{R}$ by $\Delta_a := \{z \in \mathbf{C} \mid \Re e z > a\}$.

Hereafter H^n_k, $n \in \mathbf{N}$, stands for the weighted Sobolev space associated to the norm

$$\|f\|^2_{H^n_k} := \sum_{j=0}^{n} \|\langle x\rangle^k \nabla^j f\|^2_{L^2},$$

moreover L^2_{rad} denotes the L^2 space of radially symmetric functions and $L^2_{k,j}$, $j < k$, the following space

$$L^2_{k,j} := \left\{ g \in L^2_k \ \mid \ \int x^\alpha g = 0, \ \forall\, \alpha \in \mathbf{N}^2, \ |\alpha| \le j \right\}.$$

3.1 The parabolic-elliptic case $\epsilon = 0$

Before introducing our results, we briefly recall some results on the parabolic-elliptic case $\epsilon = 0$ that we shall need in the sequel. In this case, the rescaled equation (3.2) becomes

$$\begin{cases} \partial_t g = \Delta g + \nabla(\frac{1}{2}\, x\, g - g\, \nabla v) & \text{on} \quad (0,\infty) \times \mathbf{R}^2, \\ -\Delta v = g & \text{on} \quad (0,\infty) \times \mathbf{R}^2, \end{cases} \tag{3.3}$$

and the stationary solution (G, V) to (3.3) verifies

$$\begin{aligned} \Delta G - \nabla(G\, \nabla V - \frac{1}{2} x\, G) = 0 \quad &\text{in} \quad \mathbf{R}^2, \\ \Delta V + G = 0 \quad &\text{in} \quad \mathbf{R}^2, \end{aligned}$$

with $\int_{\mathbf{R}^2} G(x)\, dx = M \in (0, 8\pi)$. The linearised equation around (G, V) is then

$$\begin{cases} \partial_t f = \Delta f + \nabla \left(\frac{1}{2} x f - f \nabla V - G \nabla u \right), \\ -\Delta u = f, \end{cases} \tag{3.4}$$

which simplifies into a single equation

$$\partial_t f = \left\{ \Delta + \nabla \left(\frac{1}{2} x f - f \nabla V - G \nabla (-\Delta)^{-1} \right) \right\} f =: \Omega f.$$

We recall the following result from [9, Section 6.1] and [15, Theorem 4.3] about properties of Ω.

Theorem 3.2. *Let $k > 7$. There exists a constant C such that*

$$\forall\, t \ge 0, \ \forall\, h \in L^2_{k,0}, \quad \|S_\Omega(t) h\|_{L^2_k} \le C\, e^{-t} \|h\|_{L^2_k},$$

hence it follows

$$\mathcal{R}_\Omega \in \mathcal{H}(\Delta_{-1}; \mathscr{B}(X_1)) \quad \text{and then} \quad \sigma(\Omega) \cap \Delta_{-1} = \emptyset,$$

where $X_1 = L^2_{rad} \cap L^2_{k,0}$.

3.2 The linearised operator

The linearised equation around the self-similar profile $(G\ ,V\ )$ associated to (3.2) is given by

$$\partial_t(f,u) = \Lambda\ (f,u) = (\Lambda_{1,}\ (f,u), \Lambda_{2,}\ (f,u)) \tag{3.5}$$

with

$$\begin{cases} \Lambda_{1,}\ (f,u) := \Delta f + \nabla(\frac{1}{2}\,x\,f - f\,\nabla V\ - G\ \nabla u), \\ \Lambda_{2,}\ (f,u) := \frac{1}{\epsilon}(\Delta u + f) + \frac{1}{2}\,x\cdot\nabla u, \end{cases}$$

The main result that we establish in this subsection is a localisation of the spectrum of Λ . We fix $k > 7$ and we introduce the Hilbert space

$$X := X_1 \times X_2, \quad X_1 := L^2_{rad} \cap L^2_{k,0} \subset L^2_{k,1}, \quad X_2 = L^2_{rad},$$

associated to the norm

$$\|(f,u)\|^2_X := \|f\|^2_{L^2_k} + \|u\|^2_{L^2}.$$

Then the following localisation of the spectrum of Λ on X holds.

Theorem 3.3. *There exists $\epsilon^* > 0$ such that in X there holds*

$$\sigma(\Lambda\ ) \cap \Delta_{-1/3} = \emptyset \quad \textit{for any} \quad \epsilon \in (0,\epsilon^*).$$

As a consequence, there exists a constant $C > 0$ such that

$$\|S_\Lambda\ (t)\|_{X\to X} \le C\,e^{-t/3} \quad \forall\, t \ge 0,\ \forall\, \epsilon \in (0,\epsilon^*).$$

The difficulty in obtaining such a result is that Λ is not a perturbation of some fixed operator Λ and we cannot apply directly the perturbation theory developed in [22, 28].

However, we can easily observe, at a formal level, that the limit system of (3.5) as $\epsilon \to 0$ is the linearised parabolic-elliptic system (3.4), which has been studied in [8, 9, 15] and for which it has been proved therein that the associated semigroup is exponentially stable in several weighted Lebesgue spaces (see Theorem 3.2). We shall then explain in the sequel why the linearised parabolic-parabolic system inherits that exponential stability for $\epsilon > 0$ small enough (which is exactly the result on Theorem 3.3).

We first obtain the following property of the spectrum of Λ on X.

Proposition 3.4. *There exist $\epsilon^*, r^* > 0$ such that in X*

$$\forall\,\epsilon \in (0,\epsilon^*) \qquad \sigma(\Lambda\ ) \cap \Delta_{-1/3} \subset \sigma_d(\Lambda\ ) \cap B(0,r^*).$$

Let us now sketch the proof of Proposition 3.4. We introduce the space

$$Y := Y_1 \times Y_2, \quad Y_1 := H^1_k \cap L^2_{k,0} \cap L^2_{rad}, \quad Y_2 := H^1 \cap L^2_{rad},$$

endowed with the norm

$$\|(f,u)\|_Y^2 := \|f\|^2_{L^2_k} + \|u\|^2_{L^2} + \|\nabla f\|^2_{L^2_k} + \|\nabla u\|^2_{L^2}.$$

We define the bounded operator $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2) : X \to X$ by

$$\mathcal{A}_1(f,u) := N(\chi_R f - \chi_1 \langle \chi_R f \rangle), \quad \mathcal{A}_2(f,u) := 0,$$

for some constants $N, R > 0$ to be chosen large enough and a smooth nonnegative radially symmetric cut-off function $\chi_R(x) := \chi(x/R)$ with $\chi \equiv 1$ on $B_{1/2}$, $\operatorname{Supp} \chi \subset B_2$ and $\langle \chi_1 \rangle = 1$, where $\langle g \rangle = \int g$.

We split the operator Λ into $\Lambda = \mathcal{A} + \mathcal{B}$ and we can prove the following properties:

- $\mathcal{A}$ is bounded from $X \to X$ and from $Y \to Y$.
- For any $a \in (-1/2, 0)$, $\mathcal{B}$ is a-hypodissipative in X and Y, in the sense that: for any $t \geq 0$

$$\|S_{\mathcal{B}}(t)\|_{X \to X} \leq C\, e^{at}, \qquad \|S_{\mathcal{B}}(t)\|_{Y \to Y} \leq C'\, e^{at}.$$

- For any $a \in (-1/2, 0)$, B has a "regularising" property, more precisely, for any $t \geq 0$,

$$\|S_{\mathcal{B}}(t)\|_{X \to Y} \leq C\, t^{-1/2}\, e^{at}.$$

Proposition 3.4 follows then from the previous properties of $\mathcal{A}$ and $\mathcal{B}$, by applying a principal spectral mapping theorem [23, Theorems 2.1] and also a version of Weyl's theorem [23, Theorems 3.1].

Another important property related to Λ is the following bound on the resolvent $\mathcal{R}_\Lambda$, which can be obtained by a perturbative argument, when $\epsilon > 0$ is small enough, using the resolvent estimates for $\mathcal{R}_\Omega$ in Theorem 3.2.

Proposition 3.5. *For any $\rho > 0$, there exists $\epsilon^*_\rho > 0$ such that in X there holds*

$$\mathcal{R}_\Lambda \in \mathcal{H}(\Delta_{-1/3} \cap B(0,\rho); \mathscr{B}(X)) \quad \textit{for any} \quad \epsilon \in (0, \epsilon^*_\rho).$$

As already mentioned above, at a formal level, the limit system of (3.5) as $\epsilon \to 0$ is the system (3.4). In particular, still at a formal level, we can see the following convergence of resolvents

$$\mathcal{R}_\Lambda(z) \to \begin{pmatrix} \mathcal{R}_\Omega(z) & 0 \\ (-\Delta)^{-1}\mathcal{R}_\Omega(z) & 0 \end{pmatrix} \quad \text{as} \quad \epsilon \to 0.$$

In fact we do not prove that the previous convergence rigorously holds, we can only prove Proposition 3.5 (which is enough to conclude).

Finally, Theorem 3.3 is a consequence of Proposition 3.4, Proposition 3.5, Theorem 3.2 and [23, Theorems 2.1].

3.3 Accurate decay estimate for the linearised operator

Define the space

$$Z := Z_1 \times Z_2, \quad Z_1 := H^1_k \cap L^2_{k,0} \cap L^2_{rad}, \quad Z_2 := H^2 \cap L^2_{rad},$$

associated to the norm

$$\|(f,u)\|^2_Z := \|f\|^2_{L^2_k} + \|u\|^2_{L^2} + \|\nabla f\|^2_{L^2_k} + \|\nabla u\|^2_{L^2} + \|\nabla^2 u\|^2_{L^2}.$$

We first obtain that the same linear stability estimate in X established in Theorem 3.3 also holds in $Z \subset X$, as stated in the following result.

Proposition 3.6. *There exists $\epsilon^* > 0$ such that there holds in Z*

$$\sigma(\Lambda\ ) \cap \Delta_{-1/3} = \emptyset, \quad \forall \epsilon \in (0, \epsilon^*).$$

As a consequence we have

$$\|S_\Lambda\ (t)\|_{Z\to Z} \le C\, e^{-t/3}, \quad \forall t \ge 0,\ \forall \epsilon \in (0,\epsilon^*).$$

This is an immediate consequence of Theorem 3.3 together with the "shrinkage extension theorem" [21, Theorem 1.1] and the following properties:

- $\mathcal{A}$ is bounded from $Z \to Z$.
- For any $a \in (-1/2, 0)$, $\mathcal{B}$ is a-hypodissipative in Z, in the sense that: for any $t \ge 0$
$$\|S_{\mathcal{B}}\ (t)\|_{X\to X} \le C\, e^{at}.$$
- For any $a \in (-1/2, 0)$, B has a "regularising" property from X to Z, more precisely, for any $t \ge 0$,
$$\|S_{\mathcal{B}}\ (t)\|_{X\to Z} \le C\, t^{-1}\, e^{at}.$$

We are now able to construct a new norm for which the semigroup is not only dissipative, but also has a stronger dissipativity property. It is important to observe that this construction will be crucial in order to deal with the nonlinear equation in next subsection.

Proposition 3.7. *Define for any $\eta > 0$ the norm*

$$|||(f,u)|||^2_Z := \eta\|(f,u)\|^2_Z + \int_0^\infty \|S_\Lambda\ (\tau)(f,u)\|^2_Z\, d\tau,$$

which is equivalent to $\|\cdot\|_Z$ thanks to Proposition 3.6. Then there exists $\eta > 0$ small enough such that the solution $S_\Lambda\ (t)(f,u)$ to the linearised equation (3.5) satisfies

$$\frac{d}{dt}|||(f,u)|||^2_Z \le -K|||(f,u)|||^2_Z - K\{\|\nabla^2 f\|^2_{L^2_k} + \|\nabla^3 u\|^2_{L^2}\} =: -K\|(f,u)\|^2_{\mathcal{Z}}, \tag{3.6}$$

for some constant $K > 0$.

The proof of Proposition 3.7 is a consequence of the exponential stability of $S_{\Lambda}\ (t)$ stablished in Proposition 3.6 together with some stronger dissipative properties of the operator $\mathcal{B}$, more precisely not only $\mathcal{B}$ is hypodissipative as explained above but we also have a gain of regularity, which implies the terms $\|\nabla^2 f\|^2_{L^2_k} + \|\nabla^3 u\|^2_{L^2}$ appearing in Proposition 3.7. When treating the nonlinear equation, these terms will be needed in order to control the loss of regularity coming from the nonlinear part (see below for more details).

3.4 The nonlinear equation

We focus now on the nonlinear parabolic-parabolic Keller-Segel system (3.2) in self-similar variables and we sketch the proof of Theorem B. The perturbation $(f,u) := (g - G\ , v - V\ )$ satisfies

$$\begin{aligned} \partial_t f &= \Lambda\ _{,1}(f,u) - \nabla \cdot (f \nabla u) \\ \partial_t u &= \Lambda\ _{,2}(f,u), \end{aligned} \tag{3.7}$$

together with the initial condition $(f,u)_{|t=0} = (f_0,u_0) := (g_0,v_0) - (G\ , V\ )$.

The proof of Theorem B can be split into three parts, and we sketch each of them below.

A priori estimate

The first and most important part is a key a priori stability estimate. In order to obtain a priori estimates on solutions to (3.7), we shall use the new dissipative norm $|||\cdot|||_Z$ constructed in Proposition 3.7. More precisely we obtain:

Lemma 3.8. *The solution (f,u) to* (3.7) *satisfies, at least formally, the following differential inequality, for some constants $C, K > 0$,*

$$\frac{d}{dt} |||(f,u)|||^2_Z \leq (C |||(f,u)|||_Z - K)\, \|(f,u)\|^2_{Z}, \tag{3.8}$$

where $\|\cdot\|_{Z}$ is defined in (3.6).

Let us sketch the proof. Denote $Q(f,u) = (-\nabla \cdot (f\nabla u), 0)$ the nonlinear part of (3.7), then we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} |||(f,u)|||^2_Z &= \eta \langle (f,u), \Lambda\ (f,u) \rangle_Z + \int_0^\infty \langle S(\tau)(f,u), S(\tau)\Lambda\ (f,u)\rangle_Z \, d\tau \\ &\quad + \eta \langle (f,u), Q(f,u)\rangle_Z + \int_0^\infty \langle S(\tau)(f,u), S(\tau) Q(f,u)\rangle_Z \, d\tau =: I_1 + I_2. \end{aligned}$$

For the first linear term, we have already obtained in Proposition 3.7 that

$$I_1 \leq -K \|(f,u)\|^2_{Z}.$$

For the second (nonlinear) term, we use the linear exponential stability of $S_\Lambda\ (t)$ in Z from Proposition 3.6 to obtain

$$\begin{aligned} I_2 &\leq \eta \|(f,u)\|_Z \, \|Q(f,u)\|_Z + \int_0^\infty \|S(\tau)(f,u)\|_Z \, \|S(\tau)Q(f,u)\|_Z \, d\tau \\ &\leq \eta \|(f,u)\|_Z \, \|Q(f,u)\|_Z + C\|(f,u)\|_Z \, \|Q(f,u)\|_Z \int_0^\infty e^{a\tau} \, e^{a\tau} \, d\tau \\ &\leq C\|(f,u)\|_Z \, \|Q(f,u)\|_Z. \end{aligned}$$

After some computations, we finally obtain that the nonlinear part $Q(f,u)$ verifies

$$\|Q(f,u)\|_Z \leq C\|(f,u)\|^2_{\mathcal{Z}},$$

and this concludes the a priori estimate of Lemma 3.8.

Regularity

We now show that starting close enough to the self-similar profile, the solution to (3.7) satisfies some strong and uniform in time estimates.

Lemma 3.9. *There is $\delta > 0$ such that, if $|||(f_0,u_0)|||_Z \leq \delta$, there exists a solution $(f,u) \in C([0,\infty);Z)$ to* (3.7) *that verifies*

$$\forall\, t \geq 0, \qquad |||(f,u)(t)|||^2_Z + \frac{K}{2}\int_0^t \|(f,u)(\tau)\|^2_{\mathcal{Z}} \, d\tau \leq 2\delta^2. \tag{3.9}$$

Indeed, at least formally, taking $\delta := K/(2C)$ in (3.8), we see that $t \mapsto |||(f,u)(t)|||_Z$ is decreasing if $|||(f_0,u_0)|||_Z \leq \delta$, and then the a priori estimate (3.9) immediately follows from (3.8). A completely rigorous proof of this is based on an iterative scheme, see e.g. [18, Theorem 5.3].

Sharp exponential convergence to the equilibrium

We end this section by presenting how we can complete the proof of Theorem B by obtaining the sharp exponential convergence for (3.7). Applying Lemma 3.8 to the solution (f,u) constructed above and using the estimate (3.9), we obtain

$$\begin{aligned} \frac{d}{dt}|||(f,u)|||^2_Z &\leq (C|||(f,u)|||_Z - K)\,\|(f,u)\|^2_{\mathcal{Z}} \\ &\leq (C\delta - K)\,\|(f,u)\|^2_{\mathcal{Z}} \leq (C\delta - K)\,C'\,|||(f,u)|||^2_Z. \end{aligned}$$

If $\delta > 0$ is small enough so that $C\delta - K \leq -K/2$, this differential inequality implies the exponential decay

$$|||(f,u)(t)|||_Z \leq e^{-\frac{K\,C'}{4}t}\,|||(f_0,u_0)|||_Z.$$

Finally, we can recover the optimal decay rate $O(e^{at})$ of the linearised semigroup in Proposition 3.6 by performing a bootstrap argument as in [18, Proof of Theorem 5.3].

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