



# Séminaire Laurent Schwartz

## EDP et applications

Année 2014-2015

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*Séminaire Laurent Schwartz — EDP et applications* (2014-2015), Exposé n° XVI, 13 p.

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## Inverse scattering without phase information

R.G. Novikov

**Abstract.** We report on nonuniqueness, uniqueness and reconstruction results in quantum mechanical and acoustic inverse scattering without phase information. We are motivated by recent and very essential progress in this domain.

### 1. Introduction

We consider the equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d \geq 1, \quad E > 0, \quad (1.1)$$

where  $\Delta$  is the Laplacian in  $x$ ,  $v$  is a coefficient, e.g. such that

$$\begin{aligned} v &\in L^\infty(\mathbb{R}^d), \quad \text{supp } v \subset D, \\ D &\text{ is an open bounded domain in } \mathbb{R}^d. \end{aligned} \quad (1.2)$$

Equation (1.1) can be considered as the quantum mechanical Schrödinger equation at fixed energy  $E$ . In quantum mechanics such an equation arises for an elementary particle interacting with a macroscopic object being in  $D$ . In this case  $v$  is the potential of this interaction.

Equation (1.1) can also be considered as the time-harmonic acoustic equation for pressure oscillations  $p = e^{i\omega t}\psi(x)$  at fixed frequency  $\omega$ . In this setting

$$v(x) = (1 - (n(x))^2)\left(\frac{\omega}{c_0}\right)^2, \quad E = \left(\frac{\omega}{c_0}\right)^2, \quad (1.3)$$

where  $n(x)$  is a scalar index of refraction, ( $n(x) \equiv 1$  on  $\mathbb{R}^d \setminus D$ ),  $c_0$  is a reference sound speed.

For equation (1.1) we consider the scattering solutions  $\psi^+ = \psi^+(\cdot, k)$  continuous on  $\mathbb{R}^d$  and specified by the following asymptotics as  $|x| \rightarrow \infty$ :

$$\begin{aligned} \psi^+(x, k) &= e^{ikx} + c(d, |k|) \frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f(k, |k| \frac{x}{|x|}) + O\left(\frac{1}{|x|^{(d+1)/2}}\right), \\ x \in \mathbb{R}^d, \quad k \in \mathbb{R}^d, \quad k^2 = E, \quad c(d, |k|) &= -\pi i (-2\pi i)^{(d-1)/2} |k|^{(d-3)/2}, \end{aligned} \quad (1.4)$$

for some a priori unknown  $f$ . The function  $f = f(k, l)$ ,  $k \in \mathbb{R}^d$ ,  $l \in \mathbb{R}^d$ ,  $k^2 = l^2 = E$ , arising in (1.4) is the scattering amplitude for equation (1.1). In connection with determining  $\psi^+$  and  $f$  from  $v$ , see, e.g. [BS], [F2], [N8], [N10] and references therein.

We recall that  $\psi^+$  describes scattering of the incident plane waves described by  $e^{ikx}$  on the scatterer described by  $v$ . In addition, the second term on the right-hand side of (1.4) describes the scattered spherical waves.

In addition, the modulus squared  $|f(k, l)|^2$  of the scattering amplitude is known as the differential scattering cross section. In quantum mechanics this modulus squared describes the probability density of scattering of particle with initial impulse  $k$  into direction  $l/|l| \neq k/|k|$ ; see, e.g. Section 6 of Chapter 1 of [FM].

Let

$$\mathbb{S}_r^{d-1} = \{k \in \mathbb{R}^d : |k| = r\}, \quad r > 0. \quad (1.5)$$

One can see that the scattering amplitude  $f$  for equation (1.1) at fixed  $E$  is defined on

$$\Omega_{f,E} = \mathbb{S}_{\sqrt{E}}^{d-1} \times \mathbb{S}_{\sqrt{E}}^{d-1}. \quad (1.6)$$

We set

$$\Omega_{f,\Lambda} = \cup_{E \in \Lambda} \Omega_{f,E}, \quad \Lambda \subseteq \mathbb{R}_+, \quad (1.7)$$

where

$$\mathbb{R}_+ = ]0, +\infty[, \quad \mathbb{R}_- = ]-\infty, 0[. \quad (1.8)$$

We start with considerations of the following inverse scattering problems for equation (1.1) under assumptions (1.2):

**Problem 1.1.** Reconstruct potential  $v$  on  $\mathbb{R}^d$  from its scattering amplitude  $f$  on some appropriate  $\Omega'_f \subseteq \Omega_{f,\Lambda}$ ,  $\Lambda = \mathbb{R}_+$ .

**Problem 1.2.** Reconstruct potential  $v$  on  $\mathbb{R}^d$  from its phaseless scattering data  $|f|^2$  on some appropriate  $\Omega'_f \subseteq \Omega_{f,\Lambda}$ ,  $\Lambda = \mathbb{R}_+$ .

Actually, Problem 1.2 is Problem 1.1 without phase information on the scattering amplitude  $f$ .

Note that in quantum mechanical scattering experiments (in the framework of model described by equation (1.1)) the complete scattering amplitude  $f$  is not accessible for direct measurements, whereas the phaseless scattering data  $|f|^2$  can be measured directly: namely,  $|f|^2$  has direct probabilistic interpretation. Therefore, Problem 1.2 is of particular interest in the framework of quantum mechanical inverse scattering.

As regards to acoustic scattering experiments (in the framework of the model described by (1.1), (1.3)), the complete scattering amplitude  $f$  can be measured directly. Nevertheless, in some cases it may be more easy to measure the phaseless scattering data  $|f|^2$ . Therefore, Problem 1.2 is also of interest in the framework of acoustic inverse scattering.

However, in the literature many more results are given on Problem 1.1 (see [ABR], [AW], [Ber], [Buc], [BAR], [ChS], [DT], [E], [F1], [F2], [GS], [G], [HH], [HN], [I], [IN], [L], [Mar], [Mel], [Mos], [New], [N1]-[N8], [NM], [R], [S] and references therein) than on Problem 1.2 (see Chapter X of [ChS] and the works [AS], [K], [KS], [KR1], [KR2], [N9], [N10], [N11] and references therein).

In particular, for the case of the Schrödinger equation the following results are known. It is well-known that for this case the following formula holds:

$$\widehat{v}(k-l) = f(k,l) + O(E^{-1/2}) \quad \text{as } E \rightarrow +\infty, \quad (k,l) \in \Omega_{f,E}, \quad (1.9)$$

where

$$\widehat{v}(p) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad p \in \mathbb{R}^d; \quad (1.10)$$

see, for example, [N8]. Actually, formula (1.9) is known as a Born formula at high energies. As a mathematical theorem formula (1.9) goes back to [F1].

Using (1.9), (1.10) one can see that for the Schrödinger equation in dimension  $d \geq 2$  the scattering amplitude  $f$  at high energies uniquely determines  $v$ . More precisely, using (1.9) for  $d \geq 2$  with

$$k = k_E(p) = \frac{p}{2} + \left(E - \frac{p^2}{4}\right)^{1/2} \gamma(p), \quad (1.11)$$

$$l = l_E(p) = -\frac{p}{2} + \left(E - \frac{p^2}{4}\right)^{1/2} \gamma(p),$$

$$|\gamma(p)| = 1, \quad \gamma(p)p = 0, \quad (1.12)$$

one can reconstruct  $\widehat{v}(p)$  from  $f$  at high energies for any  $p \in \mathbb{R}^d$ .

It is also known that, under assumptions (1.2), in dimension  $d \geq 2$ , potential  $v$  is uniquely determined by its scattering amplitude  $f$  at fixed energy; see [N1] for  $d \geq 3$  and [Buc] for  $d = 2$ .

In addition, for the Schrödinger equation, under assumptions (1.2), in dimension  $d = 1$ , the scattering amplitude  $f$  on appropriate  $\Omega'_f$  uniquely determines  $v$  due to results of [NM] based on the Gel'fand-Levitan-Marchenko theory presented in [DT], [F2], [L], [Mar]; see comments to Problem 4.1 in Section 4 for more information.

On the other hand, it is known that the phaseless scattering data  $|f|^2$  on  $\Omega_{f,\Lambda}$ ,  $\Lambda = \mathbb{R}_+$ , do not determine  $v$  uniquely, in general. In particular, the following formulas hold:

$$\begin{aligned} f_y(k, l) &= e^{i(k-l)y} f(k, l), \\ |f_y(k, l)|^2 &= |f(k, l)|^2, \quad (k, l) \in \Omega_{f,\Lambda}, \quad \Lambda = \mathbb{R}_+, \quad y \in \mathbb{R}^d, \end{aligned} \quad (1.13)$$

where  $f$  is the scattering amplitude for  $v$  and  $f_y$  is the scattering amplitude for  $v_y$ , where

$$v_y(x) = v(x - y), \quad x \in \mathbb{R}^d, \quad y \in \mathbb{R}^d; \quad (1.14)$$

see, for example, [N7], [N9].

In view of the aforementioned nonuniqueness for Problem 1.2, in our recent works [N9], [N10], [N11] we have considered some modifications of this initial phaseless inverse scattering problem; see Problems 2.1, 3.1, 3.2 and 4.2, 4.3 (of the present paper). Our results in this connection are presented in Sections 2, 3 and 4. Note that in considering appropriate modifications of Problem 1.2 we were stimulated by uniqueness results of [K] on the aforementioned Problem 3.2.

Actually, we consider Problems 2.1, 3.1, 3.2, 4.2, 4.3 of Sections 2, 3 and 4 as the most appropriate modifications of Problem 1.2. The point is that in Problems 2.1, 3.1, 3.2, 4.2, 4.3 the phaseless scattering data can be measured directly and there is no principle nonuniqueness of Problem 1.2. In addition, in connection with the most recent results on Problem 3.2 we refer to [KR2].

## 2. Results of [N9]

Let  $v$  satisfy (1.2) for some fixed  $D$  and  $w_1, \dots, w_n$  be additional a priori known background scatterers such that

$$\begin{aligned} w_j &\in L^\infty(\mathbb{R}^d), \quad \text{supp } w_j \subset \Omega_j, \\ \Omega_j &\text{ is an open bounded domain in } \mathbb{R}^d, \quad \Omega_j \cap D = \emptyset, \\ w_j &\neq 0, \quad w_{j_1} \neq w_{j_2} \text{ for } j_1 \neq j_2 \text{ in } L^\infty(\mathbb{R}^d), \end{aligned} \quad (2.1)$$

where  $j, j_1, j_2 \in \{1, \dots, n\}$ .

Let

$$S = \{|f|^2, |f_j|^2, j = 1, \dots, n\}, \quad (2.2)$$

where  $f$  is the initial scattering amplitude for  $v$ ,  $f_j$  is the scattering amplitude for

$$v_j = v + w_j, \quad j = 1, \dots, n. \quad (2.3)$$

In other words,  $S$  consists of the phaseless scattering data  $|f|^2, |f_1|^2, \dots, |f_n|^2$  (differential scattering cross sections) measured sequentially for the unknown scatterer  $v$  and then for the unknown scatterer  $v$  in the presence of known scatterer  $w_j$  nonintersecting  $v$  for  $j = 1, \dots, n$ .

In addition, we consider the following modification of Problem 1.2:

**Problem 2.1.** Reconstruct potential  $v$  on  $\mathbb{R}^d$  from the phaseless scattering data  $S$  on some appropriate  $\Omega'_f \subseteq \Omega_{f,\Lambda}$ ,  $\Lambda = \mathbb{R}_+$ , and for some appropriate background scatterers  $w_1, \dots, w_n$ .

Problem 2.1 in dimension  $d = 1$  for  $n = 1$  was, actually, considered in [AS]. However, to our knowledge, Problem 2.1 in dimension  $d \geq 2$  has not yet been considered in the literature before the recent work [N9].

We represent the Fourier transforms  $\widehat{v}$  and  $\widehat{w}_j$  as follows:

$$\widehat{v}(p) = |\widehat{v}(p)|e^{i\alpha(p)}, \quad \widehat{w}_j(p) = |\widehat{w}_j(p)|e^{i\beta_j(p)}, \quad (2.4)$$

where  $p \in \mathbb{R}^d$ ,  $j = 1, \dots, n$ .

In the next theorem we give explicit formulas for finding the Fourier transform  $\widehat{v}$  from the phaseless quantum mechanical scattering data  $S$  of (2.2) at high energies for  $d \geq 2$ ,  $n = 2$ , for appropriate background scatterers  $w_1, w_2$ .

**Theorem 2.1** ([N9]). *Suppose that complex-valued  $v$  and  $w_1, w_2$  satisfy (1.2) and (2.1), where  $d \geq 2$ . Then the following formulas hold:*

$$\begin{aligned} |\widehat{v}_j(p)|^2 &= |f_j(k, l)|^2 + O(E^{-1/2}) \text{ as } E \rightarrow +\infty, \\ p \in \mathbb{R}^d, \quad (k, l) \in \Omega_{f,E}, \quad k - l = p, \quad j = 0, 1, 2, \end{aligned} \quad (2.5)$$

where  $v_0 = v$ ,  $f_0 = f$ ,  $v_j$  is defined by (2.3),  $j = 1, 2$ . In addition,

$$\begin{aligned} |\widehat{v}| \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} &= (2 \sin(\beta_2 - \beta_1))^{-1} \times \\ &\begin{pmatrix} \sin \beta_2 & -\sin \beta_1 \\ -\cos \beta_2 & \cos \beta_1 \end{pmatrix} \begin{pmatrix} |\widehat{w}_1|^{-1}(|\widehat{v}_1|^2 - |\widehat{v}|^2 - |\widehat{w}_1|^2) \\ |\widehat{w}_2|^{-1}(|\widehat{v}_2|^2 - |\widehat{v}|^2 - |\widehat{w}_2|^2) \end{pmatrix}, \end{aligned} \quad (2.6)$$

where

$$|\widehat{v}| = |\widehat{v}(p)|, \quad \alpha = \alpha(p), \quad \beta_j = \beta_j(p), \quad |\widehat{w}_j| = |\widehat{w}_j(p)|, \quad |\widehat{v}_j| = |\widehat{v}_j(p)|, \quad j = 1, 2,$$

$$\sin(\beta_2(p) - \beta_1(p)) \neq 0, \quad |\widehat{w}_1(p)| \neq 0, \quad |\widehat{w}_2(p)| \neq 0, \quad p \in \mathbb{R}^d. \quad (2.7)$$

Formulas (2.5), (2.6) are explicit formulas for finding  $\widehat{v}$  from  $S = \{|f|^2, |f_1|^2, |f_2|^2\}$  at high energies  $E$  in dimension  $d \geq 2$  for appropriate  $w_1, w_2$ . For simplicity, in connection with (2.6), (2.7) we can assume also that

$$w_2(x) = w_1(x - y), \quad y \in \mathbb{R}^d \setminus \{0\}. \quad (2.8)$$

In this case we have

$$|\widehat{w}_2(p)| = |\widehat{w}_1(p)|, \quad \beta_2(p) = \beta_1(p) + py \pmod{2\pi}, \quad p \in \mathbb{R}^d. \quad (2.9)$$

As a corollary of (2.8), (2.9),

$$\begin{aligned} \text{conditions (2.7) are fulfilled if and only if } & p \in \mathbb{R}^d \setminus (A_y \cup Z), \\ A_y = \{p \in \mathbb{R}^d : e^{2ipy} = 1\}, \quad Z = \{p \in \mathbb{R}^d : |\widehat{w}_j(p)| = 0\}, & j = 1, 2. \end{aligned} \quad (2.10)$$

Here, the definition of  $Z$  does not depend on  $j$ .

We have, in particular, that

$$A_y \text{ is closed and } Meas A_y = 0 \text{ in } \mathbb{R}^d, \quad y \neq 0, \quad (2.11)$$

$$Z \text{ is closed and } Meas Z = 0 \text{ in } \mathbb{R}^d, \quad (2.12)$$

where properties (2.12) follow from (2.1).

Using (2.5)-(2.7) and assuming, for example, (2.8) one can see that the phaseless quantum mechanical scattering data  $S = \{|f|^2, |f_1|^2, |f_2|^2\}$  at high energies in dimension  $d \geq 2$  and the background scatterers  $w_1, w_2$  uniquely determine  $v$ . More precisely, using (2.5) with  $k, l$  as in (1.11) and then using (2.6), one can reconstruct  $\widehat{v}(p)$  from  $S$  at high energies and from  $\widehat{w}_1, \widehat{w}_2$  for any  $p \in \mathbb{R}^d \setminus (A_y \cup Z)$ , where  $A_y, Z$  are defined in (2.10) and have, in particular, properties (2.11), (2.12).

Actually, Theorem 2.1 is an analog for the phaseless case of inverse scattering results based on (1.9), (1.10).

In [N9], as a corollary of explicit formulas for solving Problem 2.1 at high energies, we give also a global uniqueness result for this problem with appropriate data on a fixed energy neighborhood.

### 3. Results of [N10]

Let  $\mathbb{S}_r^{d-1}$  be defined by (1.5) and

$$\mathcal{B}_r = \{x \in \mathbb{R}^d : |x| < r\}, \quad r > 0. \quad (3.1)$$

We suppose that

$$D \subseteq \mathcal{B}_r, \quad (3.2)$$

where  $D$  is the domain of assumptions (1.2).

As scattering data for equation (1.1), under assumptions(1.2), (3.2), we consider also  $\psi^+(x, k)$ , where  $(x, k) \in \Omega'_{\psi, E} \subseteq \Omega_{\psi, E}$ ,

$$\Omega_{\psi, E} = (\mathbb{R}^d \setminus \mathcal{B}_r) \times \mathbb{S}_{\sqrt{E}}^{d-1}, \quad (3.3)$$

where  $\psi^+$  are the scattering solutions of the introduction. In addition, in a similar way with (1.7) we set

$$\Omega_{\psi, \Lambda} = \cup_{E \in \Lambda} \Omega_{\psi, E}, \quad \Lambda \subseteq \mathbb{R}_+. \quad (3.4)$$

Actually, the function  $\psi^+$  can be also considered as scattering amplitude for scattering of spherical waves.

We recall that the function  $R^+$  describing scattering of spherical waves generated by point sources can be defined as the Schwartz kernel of the standard resolvent  $(-\Delta + v - E - i0)^{-1}$ . Thus,  $R^+ = R^+(x, x', E)$ ,  $x \in \mathbb{R}^d$ ,  $x' \in \mathbb{R}^d$ . Note that  $R^+(x, x', E) = R_0^+(|x - x'|, E)$  for  $v \equiv 0$ , where

$$R_0^+(|x|, E) = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{e^{i\xi x} d\xi}{\xi^2 - E - i0}, \quad (3.5)$$

where the right-hand side of (3.5) is spherically symmetric in  $x \in \mathbb{R}^d$ .

The function  $R^+(x, x', E)$  at fixed  $x' \in \mathbb{R}^d$  describes scattering of the spherical wave  $R_0^+(|x - x'|, E)$  generated by a point source at  $x'$ . In addition,

$$R^+(x, x', E) = -\frac{c(d, \sqrt{E})}{(2\pi)^d} \frac{e^{i\sqrt{E}|x|}}{|x|^{(d-1)/2}} \psi^+(x', -\sqrt{E} \frac{x}{|x|}) + O\left(\frac{1}{|x|^{(d+1)/2}}\right) \text{ as } |x| \rightarrow \infty \text{ at fixed } x', \quad (3.6)$$

$$R^+(x, x', E) = R^+(x', x, E), \quad (3.7)$$

where  $\psi^+$  is the function of the introduction,  $c$  is the constant in (1.4); see [N10] and Section 1 of Chapter IV of [FM].

In particular, in view of (1.4), (3.6), the function  $\psi^+$  can be considered as scattering amplitude for scattering of spherical waves.

We recall that, under assumptions (1.2), (3.2), the plane wave scattering amplitude  $f$  on  $\Omega_{f, E}$  uniquely and constructively determine  $\psi^+$  on  $\partial\mathcal{B}_r \times \mathbb{S}_{\sqrt{E}}^{d-1}$  and vice versa; see [Ber].

In addition, we consider the following modification of Problem 1.2:

**Problem 3.1.** Reconstruct potential  $v$  on  $\mathbb{R}^d$  from its phaseless scattering data  $|\psi^+|^2$  on some appropriate  $\Omega'_{\psi} \subseteq \Omega_{\psi, \Lambda}$ ,  $\Lambda = \mathbb{R}_+$ .

To our knowledge, Problem 3.1 has not yet been considered in the literature before the recent works [N10], [N11].

We represent  $f$  and  $c$  of (1.4) as follows:

$$f(k, l) = |f(k, l)|e^{i\alpha(k, l)}, \quad c(d, |k|) = |c(d, |k|)|e^{i\beta(d, |k|)}. \quad (3.8)$$

We define

$$a(x, k) = |x|^{(d-1)/2}(|\psi^+(x, k)|^2 - 1), \quad x \in \mathbb{R}^d \setminus \{0\}, \quad k \in \mathbb{R}^d \setminus \{0\}. \quad (3.9)$$

In the next theorem we give explicit formulas for finding the complex plane wave scattering amplitude  $f$  from the phaseless scattering data  $|\psi^+|^2$  for equation (1.1) at fixed  $E$  for  $d \geq 2$ .

**Theorem 3.1** (variation of Theorem 2.1 of [N10]). *Let real-valued  $v$  satisfy (1.2),  $d \geq 2$ , and  $f$ ,  $a$  be the functions of (1.4), (3.9). Then:*

$$\begin{aligned} |f| \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} &= (2|c| \sin(\tau(|k| - \frac{kx}{|x|})))^{-1} \times \\ &\left( \begin{pmatrix} \sin(|k||x| - kx + \beta + \tau(|k| - \frac{kx}{|x|})) & -\sin(|k||x| - kx + \beta) \\ \cos(|k||x| - kx + \beta + \tau(|k| - \frac{kx}{|x|})) & -\cos(|k||x| - kx + \beta) \end{pmatrix} \right) \times \\ &\begin{pmatrix} a(x, k) \\ a(x + \tau x/|x|, k) \end{pmatrix} + O(|x|^{-\sigma}), \quad |x| \rightarrow \infty, \end{aligned} \quad (3.10)$$

$$\sigma = 1/2 \text{ for } d = 2, \quad \sigma = 1 \text{ for } d \geq 3, \quad (3.11)$$

at fixed  $E > 0$ , where

$$|f| = |f(k, |k| \frac{x}{|x|})|, \quad \alpha = \alpha(k, |k| \frac{x}{|x|}), \quad |c| = |c(d, |k|)|, \quad \beta = \beta(d, |k|),$$

$$\sin(\tau(|k| - \frac{kx}{|x|})) \neq 0, \quad (3.12)$$

$k \in \mathbb{R}^d$ ,  $k^2 = E$ ,  $x \in \mathbb{R}^d$ ,  $\tau > 0$ .

Formulas (3.9)-(3.12) are explicit asymptotic formulas for finding complex  $f(k, l)$  at fixed  $(k, l) \in \Omega_{f, E}$ ,  $k \neq l$ , from  $|\psi^+(x, k)|^2$  for  $x = sl/|l|$ ,  $s \in [r_1, +\infty[$  for arbitrary large  $r_1 \geq r$  (assuming, e.g. (3.2)). These formulas give a method for reducing the phaseless inverse scattering Problem 3.1 to the well-studied inverse scattering Problem 1.1.

Actually, formulas (3.10), (3.11) follow from formulas (2.5)-(2.7), (3.11) of [N10], where (3.11) of [N10] is used for  $l/|l| = x/|x|$ ,  $s = |x|$  and for  $l/|l| = x/|x|$ ,  $s = |x| + \tau$ .

In particular, in [N10] using a version of formulas (3.10), (3.11) of the present paper we give also a global uniqueness result for Problem 3.1 at fixed energy  $E$  (i.e. for  $\Omega'_{\psi} \subseteq \Omega_{\psi, E}$ ) in dimension  $d \geq 2$ .

In addition, in [N10] using also formulas (3.6), (3.7) we give a global uniqueness result for Problem 3.2 at fixed energy  $E$  in dimension  $d \geq 2$ , where Problem 3.2 is formulated as follows.

As scattering data for equation (1.1), under assumptions (1.2), (3.2) we consider also  $R^+(x, y, E)$ , where  $(x, y) \in \Omega'_R \subseteq \Omega_R$ ,

$$\Omega_R = (\mathbb{R}^d \setminus \mathcal{B}_r) \times (\mathbb{R}^d \setminus \mathcal{B}_r), \quad (3.13)$$

where  $R^+$  is the aforementioned function describing scattering of spherical waves. In a similar way with (1.7), (3.4) we set

$$\Omega_{R,\Lambda} = \Omega_R \times \Lambda, \quad \Lambda \subseteq \mathbb{R}_+. \quad (3.14)$$

In addition, we consider the following phaseless inverse scattering problem:

**Problem 3.2.** Reconstruct potential  $v$  on  $\mathbb{R}^d$  from its phaseless scattering data  $|R^+|^2$  on some appropriate  $\Omega'_R \subseteq \Omega_{R,\Lambda}$ ,  $\Lambda = \mathbb{R}_+$ .

To our knowledge, Problem 3.2 was considered, first, in [K] at an energy interval for  $d = 3$ .

Note that in [N10] it was assumed that  $d \geq 2$ . In dimension  $d = 1$  studies of [N10] were continued in [N11].

#### 4. Results of [N11]

We consider equation (1.1) in dimension  $d = 1$ , where

$$\begin{aligned} v \text{ is real - valued, } v &\in L^1_1(\mathbb{R}), \\ v(x) &\equiv 0 \text{ for } x < 0, \end{aligned} \quad (4.1)$$

where

$$L^1_1(\mathbb{R}) = \left\{ u \in L^1(\mathbb{R}) : \int_{\mathbb{R}} (1 + |x|)|u(x)|dx < \infty \right\}. \quad (4.2)$$

For this one-dimensional equation (1.1) we consider the scattering solutions  $\psi^+ = \psi^+(\cdot, k)$  of the introduction for  $k = \sqrt{E} > 0$ . In this case, (1.4) takes the form

$$\psi^+(x, k) = \begin{cases} e^{ikx} + s_{21}(k)e^{-ikx} & \text{as } x \rightarrow -\infty, \\ s_{22}(k)e^{ikx} + o(1) & \text{as } x \rightarrow +\infty, \end{cases} \quad (4.3)$$

where

$$s_{21}(k) = -\frac{\pi i}{k} f(k, -k), \quad s_{22}(k) = 1 - \frac{\pi i}{k} f(k, k), \quad k = \sqrt{E} > 0. \quad (4.4)$$

In addition, the coefficients  $s_{21}$  and  $s_{22}$  arising in (4.3) are known as the reflection coefficient to the left and transmission coefficient to the right, respectively, for equation (1.1),  $d = 1$ .

We consider the following two types of scattering data measured on the left for the one-dimensional equation (1.1), under assumptions (4.1): (a)  $s_{21}(k)$  and (b)  $\psi^+(x, k)$ ,  $x \in X_- \subseteq \mathbb{R}_-$ , where  $k = \sqrt{E} > 0$ .

In addition, we consider the following inverse scattering problems:

**Problem 4.1.** Reconstruct potential  $v$  on  $\mathbb{R}$  from its reflection coefficient  $s_{21}$  on  $\mathbb{R}_+$ .

**Problem 4.2.** Reconstruct potential  $v$  on  $\mathbb{R}$  from its phaseless scattering data  $|\psi^+|^2$  on  $X_- \times \mathbb{R}_+$  for some appropriate  $X_-$ .

**Problem 4.3.** Reconstruct potential  $v$  on  $\mathbb{R}$  from its phaseless scattering data  $|s_{21}|^2$  on  $\mathbb{R}_+$  and  $|\psi^+|^2$  on  $X_- \times \mathbb{R}_+$  for some appropriate  $X_-$ .

One can see that Problem 4.1 is a particular case of the well-studied Problem 1.1 for  $d = 1$  of the introduction and Problem 4.2 is a particular case of Problem 3.1 for  $d = 1$  of Section 3.

For Problem 4.1 global uniqueness and reconstruction results were given in [NM] on the basis of the Gelfand-Levitan-Marchenko theory; see also [AW], [GS] and references given in [AW].

To our knowledge, Problems 4.2 and 4.3 have not yet been considered in the literature before the recent work [N11].

We use the following notations:

$$S_1(x_1, x_2, k) = \{|s_{21}(k)|^2, |\psi^+(\cdot, k)|^2 \text{ on } X_-\}, \quad (4.5)$$

where  $X_- = \{x_1, x_2 \in \mathbb{R}_- : x_1 \neq x_2\}$ ,  $k \in \mathbb{R}_+$ ;

$$S_2(x_1, x_2, x_3, k) = |\psi^+(\cdot, k)|^2 \text{ on } X_-, \quad (4.6)$$

where  $X_- = \{x_1, x_2, x_3 \in \mathbb{R}_- : x_i \neq x_j \text{ if } i \neq j\}$ ,  $k \in \mathbb{R}_+$ ;

$$S_3(x, k) = \{|\psi^+(x, k)|^2, \frac{d|\psi^+(x, k)|^2}{dx}\}, \quad x \in \mathbb{R}_-, \quad k \in \mathbb{R}_+. \quad (4.7)$$

We represent  $s_{21}$  of (4.3) as follows

$$s_{21}(k) = |s_{21}(k)|e^{i\alpha(k)}, \quad k \in \mathbb{R}_+. \quad (4.8)$$

We consider

$$a(x, k) = |\psi^+(x, k)|^2 - 1, \quad x \in \mathbb{R}_-, \quad k \in \mathbb{R}_+. \quad (4.9)$$

In the next theorem we give explicit formulas for finding complex reflection coefficient  $s_{21}(k)$  from the phaseless scattering data  $S_1(x_1, x_2, k)$  for fixed  $x_1, x_2$  and  $k$ , where  $x_1 \neq x_2 \pmod{\pi(2k)^{-1}}$ .

**Theorem 4.1** ([N11]). *Let  $v$  satisfy (4.1) and  $s_{21}, a$  be the function of (4.3), (4.9). Let  $x_1, x_2 \in \mathbb{R}_-, k \in \mathbb{R}_+, x_1 \neq x_2 \pmod{\pi(2k)^{-1}}$ . Then:*

$$|s_{21}(k)| \begin{pmatrix} \cos \alpha(k) \\ \sin \alpha(k) \end{pmatrix} = (2 \sin(2k(x_2 - x_1)))^{-1} \times \begin{pmatrix} \sin(2kx_2) & -\sin(2kx_1) \\ -\cos(2kx_2) & \cos(2kx_1) \end{pmatrix} \begin{pmatrix} a(x_1, k) - |s_{21}(k)|^2 \\ a(x_2, k) - |s_{21}(k)|^2 \end{pmatrix}. \quad (4.10)$$

Actually, Theorem 4.1 is an one-dimensional analog of Theorem 3.1.

Formulas (4.9), (4.10) give a reduction of Problem 4.3 to the well-studied Problem 4.1.

In the next theorem we give explicit formulas for finding complex reflection coefficient  $s_{21}(k)$  from the phaseless scattering data  $S_2(x_1, x_2, x_3, k)$  for fixed  $x_1, x_2, x_3$  and  $k$ , where  $x_i \neq x_j \pmod{\pi k^{-1}}$  if  $i \neq j$ .

**Theorem 4.2** ([N11]). *Let  $v$  satisfy (4.1) and  $\psi^+, s_{21}$  be the functions of (4.3). Let  $x_1, x_2, x_3 \in \mathbb{R}_-, k \in \mathbb{R}_+, x_i \neq x_j \pmod{\pi k^{-1}}$  if  $i \neq j$ . Then:*

$$|s_{21}(k)| \begin{pmatrix} \cos \alpha(k) \\ \sin \alpha(k) \end{pmatrix} = (8 \sin(k(x_2 - x_3)) \sin(k(x_2 - x_1)) \sin(k(x_1 - x_3)))^{-1} \times \\ \begin{pmatrix} \sin(2kx_3) - \sin(2kx_1) & -\sin(2kx_2) + \sin(2kx_1) \\ -\cos(2kx_3) + \cos(2kx_1) & \cos(2kx_2) - \cos(2kx_1) \end{pmatrix} \times \\ \begin{pmatrix} |\psi^+(x_2, k)|^2 - |\psi^+(x_1, k)|^2 \\ |\psi^+(x_3, k)|^2 - |\psi^+(x_1, k)|^2 \end{pmatrix}. \quad (4.11)$$

In the next theorem we give explicit formulas for finding complex reflection coefficient  $s_{21}(k)$  from the phaseless scattering data  $S_3(x, k)$  for fixed  $x$  and  $k$ .

**Theorem 4.3** ([N11]). *Let  $v$  satisfy (4.1) and  $\psi^+, s_{21}$  be the functions of (4.3). Then:*

$$\operatorname{Re}(s_{21}(k)e^{-ikx}) = -1 + (|\psi^+(x, k)|^2 - |\operatorname{Im}(s_{21}(k)e^{-ikx})|^2)^{1/2}, \\ \operatorname{Im}(s_{21}(k)e^{-ikx}) = \frac{1}{4k} \frac{d|\psi^+(x, k)|^2}{dx}, \quad (4.12)$$

where  $x \in \mathbb{R}_-, k \in \mathbb{R}_+$ , and  $(\cdot)^{1/2} > 0$ .

One can see that formulas (4.11), (4.12) give reductions of Problem 4.2 to the well-studied Problem 4.1.

In [N11], under assumptions (4.1), as corollaries of Theorems 4.1, 4.2, 4.3 and the aforementioned results of [NM], we give, in particular, global uniqueness and reconstruction results (1) for finding  $v$  on  $\mathbb{R}$  from  $S_1(x_1, x_2, \cdot)$  on  $\mathbb{R}_+$  for fixed  $x_1, x_2$ , (2) for finding  $v$  on  $\mathbb{R}$  from  $S_2(x_1, x_2, x_3, \cdot)$  on  $\mathbb{R}_+$  for fixed  $x_1, x_2, x_3$ , and (3) for finding  $v$  on  $\mathbb{R}$  from  $S_3(x, \cdot)$  on  $\mathbb{R}_+$  at fixed  $x$ .

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