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ASYMPTOTIC STABILITY OF
ZAKHAROV-KUZNETSOV SOLITONS

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ABSTRACT. In this report, we review the proof of the asymptotic stability of the Zakharov-Kuznetsov solitons in dimension two. Those results were recently obtained in a joint work with Raphaël Côte, Claudio Muñoz and Gideon Simpson.

1. INTRODUCTION

This report describes a recent work of the author with Raphaël Côte, Claudio Muñoz and Gideon Simpson [4] on the Zakharov-Kuznetsov (ZK) equation

$$(1.1) \qquad \partial_t u + \partial_{x_1} \Delta u + \partial_{x_1} u^2 = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}^d, \quad x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{d-1}.$$

Here, u denotes a real valued function. When the spatial dimension d is equal to 1, equation (1.1) becomes the well-known Korteweg- de Vries (KdV) equation. We also refer to Chapter 4 of Raphaël Côte's HDR [3] for another nice report on the subject.

The ZK equation was introduced by Zakharov and Kuznetsov in [13] to describe the propagation of ionic-acoustic waves in uniformly magnetized plasma in the two dimensional and three dimensional cases. Lannes, Linares and Saut [15] carried out the derivation of ZK from the Euler-Poisson system with magnetic field in the long wave limit and Han-Kwan [10] derived rigorously ZK from the Vlasov-Poisson system in a combined cold ions and long wave limit.

The mass and the energy

$$(1.2) \qquad M(u) = \int u^2 \, dx \quad \text{and} \quad H(u) = \int \frac{1}{2} |\nabla u|^2 - \frac{1}{3} u^3 \, dx$$

are conserved by the flow of ZK. Moreover, the equation is L^2 subcritical in dimension 2 and 3.

The well-posedness theory for ZK has been extensively studied in the recent years. Let us focus on the two and three dimensional cases:

- In the two dimensional case, Faminskii proved that the Cauchy problem associated to the ZK equation is globally well-posed in the energy space $H^1(\mathbb{R}^2)$ [7] (the local well-posedness result was pushed down to $H^s(\mathbb{R}^2)$ for $s > 3/4$ by Linares and Pastor [17] and for $s > 1/2$ by Grünrock and Herr [9] and Molinet and the author [23]).

- The best result for the ZK equation in the three dimensional case was obtained last year by Ribaud and Vento [25]. They proved local well-posedness in $H^s(\mathbb{R}^3)$ for $s > 1$. Those solutions were extended globally in time in [23].

Note however that it is still an open problem to obtain well-posedness for ZK in the energy space $H^1(\mathbb{R}^3)$.

In order to understand better the dynamic of ZK, we look for special solutions on the form

$$(1.3) \qquad u(x, t) = Q_c(x_1 - ct, x_2) \quad \text{with} \quad Q_c(x) \xrightarrow[|x| \to +\infty]{} 0 \,,$$

where $Q_c(x) = cQ(c^{1/2}x)$. Then Q must solve the elliptic PDE

$$(1.4) \qquad\qquad -\Delta Q + Q - Q^2 = 0 \,,$$

which also appears in the Nonlinear Schrödinger (NLS) equation context and was already extensively studied (see for example [1], [14]). In particular in dimension $d = 2$ and $d = 3$, (1.4) has a unique (up to translation) positive radial solution $Q \in H^1(\mathbb{R}^d)$ that we will call *ground state* solution. It is also well known that $Q \in C^\infty(\mathbb{R})$ and satisfies the pointwise decay estimates

$$\partial^\alpha Q(x) \lesssim e^{-|x|}, \qquad x \in \mathbb{R}^d, \qquad \alpha \in \mathbb{N}^d \,.$$

Remark 1.1. We proved in [4] that (1.1) has no other finite energy solutions of the form $\phi(x_1 - c_1 t, x_2 - c_2 t)$ with $c_2 = 0$.

The solutions of (1.1) of the form (1.3) with $Q = Q$ are called *solitary waves* or *solitons*. They were proved by de Bouard [5] to be orbitally stable in $H^1(\mathbb{R}^d)$ for $d = 2$ and 3. Our goal here is to prove that they are actually asymptotically stable in $H^1(\mathbb{R}^d)$ by following the Martel, Merle approach [19, 20, 21]. Our main result writes as follows.

Theorem 1.1 (Asymptotic stability. Côte, Muñoz, P., Simpson [4])**.** *Assume $d = 2$. Let $c_0 > 0$. For any $\gamma > 0$, there exists $\delta_0 > 0$ such that if $0 < \delta \leq \delta_0$ and $u \in C(\mathbb{R} : H^1(\mathbb{R}^2))$ is a solution of (1.1) satisfying*

$$(1.5) \qquad\qquad \|u_0 - Q\|_{H^1} \leq \delta \,,$$

then the following holds true.
There exist $c_+ > 0$ with $|c_+ - c_0| \leq K_0 \delta$, for some positive constant K_0 independent of δ_0, and $\rho = (\rho_1, \rho_2) \in C^1(\mathbb{R} : \mathbb{R}^2)$ such that

$$(1.6) \qquad u(\cdot, t) - Q_{c_+}(\cdot - \rho(t)) \xrightarrow[t \to +\infty]{} 0 \quad in \ H^1(x_1 > \gamma t) \,,$$

$$(1.7) \qquad\qquad \rho_1'(t) \xrightarrow[t \to +\infty]{} c_+ \quad and \quad \rho_2'(t) \xrightarrow[t \to +\infty]{} 0 \,.$$

Let us start with a few remarks.

Remark 1.2. The convergence in (1.6) can not hold in the whole space $H^1(\mathbb{R}^2)$, because of conservation of mass and energy, and of the variational characterization of the soliton. There must be some loss, which can be due to smaller (and slower) solitons, or to dispersion. One way to get rid of them is to use weighted spaces. Our analysis here is sharper, as we use local spaces without weights.
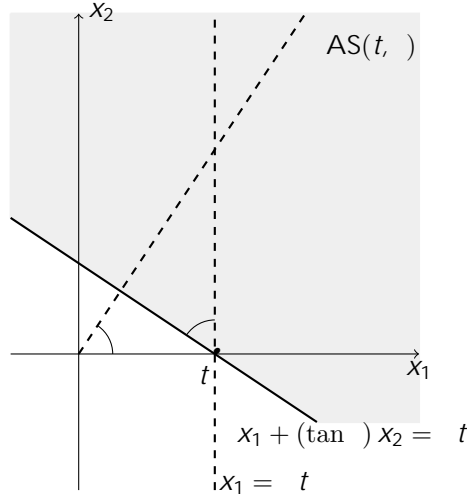
FIGURE 1. $\mathsf{AS}(t,\theta) := \left\{ (x_1, x_2) \in \mathbb{R}^2 \;:\; x_1 - \beta t + (\tan\theta)\, x_2 > 0 \right\}$.

Remark 1.3. When looking at plane wave solutions of the linear part of (1.1) on the form $u(x_1, x_2, t) = e^{i(k_1 x_1 + k_2 x_2 - wt)}$ with $w(k_1, k_2) = -(k_1^3 + k_1 k_2^2)$, we find out that the *group velocity* vector $\nabla w = -\left(3k_1^2 + k_2^2, 2k_1 k_2\right)^T$ is always contained in a semi-cone of angle $\frac{\pi}{3}$ around the negative k_1 direction. In particular dispersion points toward left while solitons evolve on the right. This is in sharp contrast with other famous multidimensional models such as the NLS and KP1 equations, for which asymptotic stability is still a very challenging open problem.

Remark 1.4. It will be clear from the proof that the convergence in (1.6) can also be obtained in regions of the form

$$\mathsf{AS}(t,\theta) := \left\{ (x_1, x_2) \in \mathbb{R}^2 \;:\; x_1 - \beta t + (\tan\theta)\, x_2 > 0 \right\}, \quad \text{where} \quad \theta \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right).$$

Note that the maximal angle of improvement θ_0 must be strictly less than $\frac{\pi}{3}$ on each side of the vertical line $x_1 = \beta t$ (see Figure 1).

Remark 1.5. Our proof does not rely on the structure of the nonlinearity of (1.1) (*i.e.* $\partial_{x_1}(u^2)$) neither on the dimension d. Actually, our main theorem could be extended to (1.1) in dimension $d = 3$ or to the following generalization of gZK

$$(1.8) \qquad \qquad \partial_t u + \partial_{x_1}(\Delta u + |u|^{p-1} u) = 0,$$

where p is a real number $1 < p < 1 + \frac{4}{d}$ under the following conditions:
- The Cauchy problem associated to (1.1) with $d = 3$ or to (1.8) is well-posed in H^1.
- The spectral condition $\langle L^{-1}\Lambda Q, \Lambda Q \rangle < 0$[1] holds true.

This spectral condition was shown in the appendix of [4] to be true in dimension $d = 2$ for $2 \leq p < p_2$, where p_2 is a real number satisfying $2 < p_2 < 3$.

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[1] Here L denotes the operator linearizing (1.4) around Q. See below for a precise definition in the case $p = 2$.

On the other hand, in dimension $d = 3$, it is shown that $\langle \mathsf{L}^{-1}\Lambda Q, \Lambda Q \rangle > 0$. Note however that in this case, one could try to verify the more general property: the operator L restricted to the space $\{\ker \mathsf{L}, \Lambda Q\}^{\perp}$ is positive definite.

Recall that the first result of asymptotic stability of solitons for generalized KdV equations was proved by Pego and Weinstein [24] in weighted spaces. In [19], Martel and Merle have given the first asymptotic result for the solitons of gKdV in the energy space H^1. They improved their result in [20] and generalized it to a larger class of nonlinearities than the pure power case in [21].

Their proof relies on a Liouville type theorem for L^2-compact solutions around a soliton. Then, it is proved that a solution near a soliton converges (up to subsequence) to a limit object, whose emanating solution satisfies a good decay property. Due to the rigidity result, this limit object has to be a soliton. One of the main ingredient in the proofs is the use of monotonicity formulas for a part of the mass and the energy.

It is worth noting that this technique of proof was also adapted to prove asymptotic stability in the energy space for other one dimensional models such as the Benjamin-Bona-Mahony equation [6], the Benjamin-Ono equation [12] and the Gross-Pitaevskii equation [2, 8].

Here, we adapt the ideas of Martel and Merle to a multidimensional model. As far as we know, Theorem 1.1 is the first result of asymptotic stability for a two dimensional model, in the energy space, and with no nonstandard spectral assumptions on the linearized dynamical operator.

In the next section, we state the Liouville theorems and explain what are the main new difficulties in proving them. In Section 3, we give a sketch of the proof of Theorem 1.1, while Section 4 is dedicated to the stability of multi-solitons.

2. Liouville theorems

The Liouville theorems classify the L^2 compact solutions (solutions which are localized in the x_1 direction, uniformly in x_2) around the solitons.

2.1. **Linear Liouville theorem.** First, we prove such a property in the linear case. Let L_c denote the linearized operator of (1.4) around Q_c, *i.e.*

$$\mathsf{L}_c = -\Delta + c - 2Q_c\,.$$

In the case $c = 1$, we also denote $\mathsf{L} = \mathsf{L}_1$.

The spectral properties of the operator L are now well understood (see for example [26]).

- L is a self-adjoint operator and $\sigma_{ess}(\mathsf{L}) = [1, +\infty)$.
- $\ker \mathsf{L} = \operatorname{span}\{\partial_{x_1}Q, \partial_{x_2}Q\}$.
- L has a unique single negative eigenvalue $-\lambda_0$ (with $\lambda_0 > 0$).
- Let Λ denote the scaling operator, *i.e.*

$$\Lambda Q := \left.\frac{d}{dc}Q_c\right|_{c=1} = Q + \frac{1}{2}x \cdot \nabla Q.$$

Then, $\mathsf{L}\Lambda Q = -Q$ and $\langle Q\Lambda Q \rangle = \frac{1}{2}\|Q\|_{L^2}^2$.

Theorem 2.1 (Linear Liouville property around Q_{c_0}. Côte, Muñoz, P., Simpson [4])**.**
Let $c_0 > 0$ and $\psi \in C(\mathbb{R} : H^1(\mathbb{R}^2))$ be a solution to

(2.1) $$\psi_t = \partial_{x_1} L_{c_0} \psi \quad \text{on } \mathbb{R} \times \mathbb{R}^2.$$

Moreover, assume that there exists a constant $\gamma > 0$ such that

(2.2) $$\int_{x_2} \psi^2(x_1, x_2, t)\, dx_2 \lesssim e^{-\gamma|x_1|}, \quad \forall (x_1, t) \in \mathbb{R}^2.$$

Then, there exists $(a_1, a_2) \in \mathbb{R}^2$ such that

(2.3) $$\psi(x, t) = a_1 \partial_{x_1} Q_{c_0}(x) + a_2 \partial_{x_2} Q_{c_0}(x), \quad \forall (x, t) \in \mathbb{R}^3.$$

Remark 2.1. We can replace assumption (2.2) by the weaker assumption that the solution ψ is *L^2-compact* in the x_1 direction, *i.e.*: $\psi \in C_b(\mathbb{R} : H^1(\mathbb{R}^2))$ and

$$\forall \epsilon > 0, \quad \exists A > 0 \text{ such that } \sup_{t\in\mathbb{R}} \int_{|x_1|>A} \psi^2(x, t)\, dx \le \epsilon.$$

Sketch of the proof of Theorem 2.1. Without loss of generality, we can work with $c_0 = 1$.

1) Following Martel [18] and Martel, Merle [21], we work on *the dual problem*

(2.4) $$v = L\psi + \mu_0 Q.$$

By using the equation (2.1), we verify easily that

(2.5) $$\partial_t v = L \partial_{x_1} v + \mu_0 L \partial_{x_1} Q = L \partial_{x_1} v,$$

and v satisfies the orthogonality conditions

(2.6) $$(v, \partial_{x_1} Q) = (v, \partial_{x_2} Q) = 0.$$

and

(2.7) $$(v, \Lambda Q) = 0.$$

Indeed, by using the properties of L, we have that

$$\frac{d}{dt}(v, \Lambda Q) = (L \partial_{x_1} v, \Lambda Q) = -(v, \partial_{x_1} L\Lambda Q) = (v, \partial_{x_1} Q) = 0,$$

which implies the orthogonality relation in (2.7) if μ_0 is chosen properly, since $(\Lambda Q, Q) > 0$ in the L^2-subcritical case.

This last orthogonality condition was not the one used by Martel and Merle in the gKdV context. However, the additional dimension makes things harder by inducing transversal variations that seem to destroy any virial-type inequality with a weight function depending on the extra dimension. As we will see below, the condition (2.7) seems quite natural when the weight function depends only on the x_1 variable. Moreover, it is worth noting that this condition has already been used by Kenig and Martel in the Benjamin-Ono context [12] for different reasons.

2) By using monotonicity formulas[2] for a part of the mass and the energy, we deduce *regularity properties* for the L^2-compact solution ψ. In particular $v \in C(\mathbb{R} : H^1(\mathbb{R}^2))$ and

(2.8) $$\int_{x_2} v^2(x, t)\, dx_2 \lesssim e^{-\tilde\gamma|x_1|}, \quad \forall x_1, t \in \mathbb{R},$$

for some $\tilde\gamma > 0$.

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[2]We will detail those formulas in the next section.

3) <u>Virial identity</u>. Let $\varphi \in C^2(\mathbb{R})$ be an even positive function such that $\varphi' \geq 0$ on $\mathbb{R}_+$,

$$\varphi'_{|[0,1]} = 1, \quad \varphi'(x_1) = e^{-x_1} \text{ on } [2,+\infty), \quad e^{-x_1} \leq \varphi'(x_1) \leq 3e^{-x_1} \text{ on } \mathbb{R}_+.$$

$$|\varphi''(x_1)| \leq C\varphi'(x_1) \quad \text{and} \quad |\varphi'''(x_1)| \leq C\varphi'(x_1),$$

for some positive constant C. Let Φ be defined by $\Phi(x_1) = \int_0^{x_1} \varphi(y)\,dy$. For a parameter A (to be fixed large enough), we set $\varphi_A(x_1) = A\varphi(x_1/A)$ so that $\varphi_A(x_1) = \varphi(x_1/A) =: \psi_A(x_1)$ and $\varphi_A(x_1) = x_1$ on $[-A, A]$.

Then, we have from (2.5) that

$$-\frac{1}{2}\frac{d}{dt}\int \varphi_A v^2\,dx$$

$$= \int \psi_A(\partial_{x_1}v)^2\,dx + \frac{1}{2}\int \varphi_A'(|\nabla v|^2 + v^2 - 2Qv^2)\,dx - \frac{1}{2}\int \Delta\psi_A v^2 - \int \psi_A \partial_{x_1}Qv^2\,dx.$$

On the one hand, it follows from the properties of Q and the weight function ψ_A that

$$-\int \psi_A \partial_{x_1}Q v^2\,dx \gtrsim \delta_0^3.$$

On the other hand, there exists $\gamma > 0$ such that

$$\frac{1}{2}\int \varphi_A'(|\nabla v|^2 + v^2 - 2Qv^2)\,dx \geq \gamma \int \varphi_A' v^2\,dx$$

if A is chosen large enough, assuming the *spectral property*: $\mathsf{L}_{/_{\{Q,\nabla Q\}^\perp}} > 0$. Thus, it follows that

$$(2.9) \qquad -\frac{1}{2}\frac{d}{dt}\int \varphi_A v^2\,dx \geq \frac{\gamma}{2}\int \varphi_A' v^2\,dx.$$

Therefore, we deduce by integrating (2.9) in time and using (2.8) that $v \equiv 0$, which concludes the proof of Theorem 2.1.

4) Finally, let us say a word about the *spectral property* $\mathsf{L}_{/_{\{Q,\nabla Q\}^\perp}} > 0$. From the work of Weinstein [26], it suffices to verify the property

$$(2.10) \qquad\qquad (\mathsf{L}^{-1}\Lambda Q, \Lambda Q) < 0.$$

Unlike in the gKdV context, obtaining a direct proof of this result in dimension 2 seems far from trivial because the soliton Q, and therefore the function $\mathsf{L}^{-1}\Lambda Q$, have no closed and explicit forms. This is another of the main differences with respect to the one dimensional case: here we work with a solitary wave which is not explicit at all.

Nevertheless, we were able to verify this property numerically in dimension 2 (but not in dimension 3). The graphics below represent the numerical computation of the scalar product $(\mathsf{L}^{-1}\Lambda Q, \Lambda Q)$ as a function of p, where L is the linearized operator corresponding to (1.8).

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[3]This is the advantage to work with the dual problem.

$$d = 1 \qquad d = 2 \qquad d = 3$$
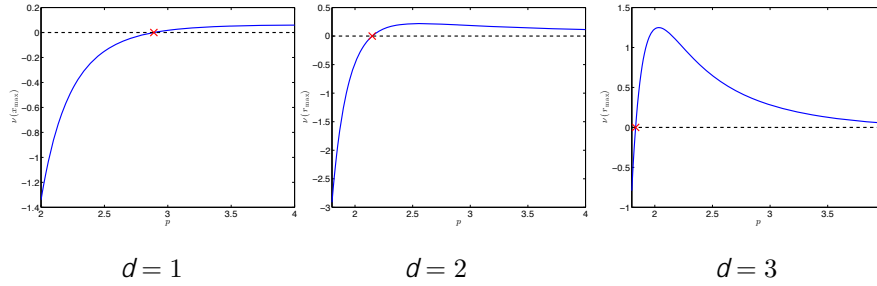
2.2. **Nonlinear Liouville theorem.** The proof of Theorem 1.1 is based on the following rigidity result for the solutions of (1.1) in spatial dimension $d = 2$ around the soliton Q_{c_0} which are uniformly localized in the direction x_1.

Theorem 2.2 (Nonlinear Liouville property around Q_{c_0}). *Assume $d = 2$. Let $c_0 > 0$. There exists $_0 > 0$ such that if $0 < \quad _0$ and $u \quad C(\mathbb{R} : H^1(\mathbb{R}^2))$ is a solution of (1.1) satisfying for some function $(t) = \quad _1(t), \quad _2(t)$ and some positive constant*

$$(2.11) \qquad \qquad u(\cdot + \ (t)) - Q_{c_0} \ _{H^1} \qquad , \qquad t \quad \mathbb{R} \,,$$

and

$$(2.12) \qquad \qquad _{x_2} u^2(x_1 + \ _1(t), x_2 + \ _2(t), t)\, dx_2 \quad e^{- \ /x_1/}, \qquad (x_1, t) \quad \mathbb{R}^2 \,,$$

then, there exist $c_1 > 0$ (close to c_0) and $^0 = (\ _1^0, \ _2^0) \quad \mathbb{R}^2$ such that

$$(2.13) \qquad \qquad u(x_1, x_2, t) = Q_{c_1}(x_1 - c_1 t - \ _1^0, x_2 - \ _2^0) \,.$$

Remark 2.2. Due to the stability result of de Bouard [5], Theorem 2.2 still hold true if we assume that

$$(2.14) \qquad \qquad u_0 - Q_{c_0} \ _{H^1} \qquad ,$$

instead of (2.11).

Remark 2.3. Theorem 2.2 still holds true if we replace assumption (2.12) by the weaker assumption that the solution u is L^2*-compact* in the x_1 direction, *i.e.*:

$$> 0, \quad A > 0 \text{ such that } \sup_{t \ \mathbb{R}} \int_{/x_1/>A} u^2(x + \ (t), t)\, dx \qquad .$$

To prove this nonlinear version of the Liouville theorem, we cannot use its linear counterpart as a black box, due to the rigidity nature of the theorem. However, the method of proof is robust and carries over very well to the nonlinear case. The additional ingredient is the modulation of the scaling parameter to make up for $_0$ in (2.4) and recover the orthogonality condition (2.7).

Moreover, we would like to emphasize another new technical difficulty appearing in the proof of Theorem 2.2 due to the extra dimension. In $\mathbb{R}^2$, we lack L^∞ control on the solutions assuming only a priori H^1 bounds. Such a control would be useful in order to ensure regularity and pointwise exponential decay for compact solutions around a soliton. Therefore, we must prove new monotonicity properties at higher regularity level, which are obtained by performing an induction on the level of

regularity and using suitable Sobolev embeddings of the form $H^1(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$, for $p < \infty$, to control the nonlinear terms.

3. PROOF OF THEOREM 1.1

In this section, we give a sketch of the proof of Theorem 1.1. Without loss of generality, we can assume that $c_0 = 1$.

1) By using de Bouard's stability result, we deduce from (1.5) that

$$(3.1) \qquad \sup_{t \in \mathbb{R}} \| u(\cdot + \rho(t), t) - Q \|_{H^1} \leq K_0 ,$$

for some positive constant $K_0 > 0$ and C^1 function $\rho = (\rho_1, \rho_2) \in C^1(\mathbb{R} : \mathbb{R}^2)$.

2) Let $t_n \to +\infty$ be a sequence such that

$$(3.2) \qquad u(\cdot + \rho(t_n), t_n) \rightharpoonup \tilde{u} \qquad \text{weak in } H^1,$$

for some $\tilde{u} \in H^1(\mathbb{R}^2)$. Let $\tilde{u}$ denote the solution of (1.1) satisfying $\tilde{u}(\cdot, 0) = \tilde{u}$. The main point is to show that this limit object enjoys nice localization properties. In particular, we prove that

$$(3.3) \qquad \| \tilde{u}(\cdot + \tilde{\rho}(t), t) - Q \|_{H^1} \leq K_0 ,$$

$$(3.4) \qquad \int_{x_2} \tilde{u}^2(x + \tilde{\rho}(t), t) dx_2 \lesssim e^{-\tilde{\gamma}|x_1|}, \quad x_1, t \in \mathbb{R},$$

for some constant $\tilde{\gamma} > 0$ and some C^1 function $\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2) \in C^1(\mathbb{R} : \mathbb{R}^2)$, and that

$$(3.5) \qquad u(\cdot + \rho(t_{n_k}), t_{n_k}) \to \tilde{u} \qquad \text{in } H^1(x_1 > -A),$$

for any $A > 0$.

Therefore, we deduce from Theorem 2.2 that $\tilde{u} = Q_{c_+}$, for some c_+ close to 1, which concludes the proof of Theorem 1.1.

3) <u>Monotonicity properties</u>: This is the main ingredient in order to prove (3.4) and (3.5).

Lemma 3.1 (Monotonicity formula in the x_1-direction for a part of the mass). *Let $\tilde{x}_1 := x_1 - \rho_1(t_0) + \frac{1}{2}(t_0 - t) - y_0$, $\psi_M(x_1) := \frac{2}{\pi} \arctan(e^{x_1/M})$ with $M \geq 4$ and*

$$(3.6) \qquad I_{y_0, t_0}(t) := \int u^2(x, t) \psi_M(\tilde{x}_1) dx.$$

Then

$$(3.7) \qquad I_{y_0, t_0}(t_0) - I_{y_0, t_0}(t) \lesssim e^{-y_0/M},$$

for all $y_0 > 0$, $t_0 \in \mathbb{R}$ and $t \leq t_0$.

Note that I_{y_0, t_0} represents the part of the mass located on the right of the soliton (which is moving here with a velocity ρ_1' close to $c_0 = 1$). Formula (3.7) tells us that the increase of this quantity is at best very small (if y_0 is chosen very large). This corresponds to the idea that some mass is expelled on the left under the form of slower and smaller solitons or dispersion.

The proof of estimate (3.7) goes in the spirit of Kato [11]. We derive I_{y_0, t_0} with respect to time, use equation (1.1), integrate by parts and use (3.1) and the Sobolev embedding $H^1(\mathbb{R}^2) \hookrightarrow L^3(\mathbb{R}^2)$ to deal with the nonlinear terms.

As a first consequence, we deduce by using (3.7) between 0 and t_0 that

$$\limsup_{t \to +\infty} \int u^2(x + \rho(t), t)\, \psi_M(x_1 - y_0)\,dx \lesssim e^{-y_0/M},$$

for any $y_0 > 0$. This provides the exponential decay on the right of u_∞. In one dimension, this fact together with (3.2) would also be sufficient to prove the L^2 part of the strong convergence on the right in (3.5). However, this argument fails to work directly in $\mathbb{R}^2$, since the strip $-A < x_1 < B$ is not compact anymore (for any $A > 0$, $B > 0$).

To solve this problem, we need to derive monotonicity formulas for the part of mass along oblique lines on the form $x_2 = \tan(\theta_0) x_1$ for $|\theta| < \frac{\pi}{3}$.

Lemma 3.2 (Monotonicity formula in oblique lines for a part of the mass). *Let $\theta_0 \in (-\frac{\pi}{3}, \frac{\pi}{3})$, $M \geq 4$ and*

$$I_{y_0, t_0, \theta_0}(t) = \int u^2(x, t)\, \psi_M\left(x_1 + (\tan \theta_0) x_2 - \rho_1(t_0) + \frac{1}{2}(t_0 - t) - y_0\right) dx.$$

Then,

$$I_{y_0, t_0, \theta_0}(t_0) - I_{y_0, t_0, \theta_0}(t) \lesssim e^{-y_0/M},$$

for every $y_0 > 0$, $t_0 \in \mathbb{R}$ and $t \leq t_0$, and

$$(3.8) \qquad \limsup_{t \to +\infty} \int u^2(x + \rho(t), t)\, \psi_M(x_1 + (\tan \theta_0) x_2 - y_0)\,dx \lesssim e^{-y_0/M},$$

for every $y_0 > 0$.

No monotonicity property seems to hold in the x_2-direction alone, mainly because the conjectured existence of small solitons moving to the right in x_1, but without restriction on the x_2 coordinate.

Lemma 3.2 is reminiscent of the geometrical properties of the ZK equation: namely, ZK behaves as a KdV equation in the x_1-direction and as a Schrödinger equation in the x_2-direction. This property is also linked to Remark 1.4.

4) <u>Strong convergence in L^2 on the right.</u> Let $A > 0$. Then, thanks to (3.8), for any $\varepsilon > 0$, there exists $R_{\frac{\varepsilon}{4}} > 0$ such that

$$\limsup_{n \to +\infty} \left\| u(\cdot + \rho(t_n), t_n) \right\|_{L^2(x_1 + x_2 > R_{\frac{\varepsilon}{4}})} + \left\| u(\cdot + \rho(t_n), t_n) \right\|_{L^2(x_1 - x_2 > R_{\frac{\varepsilon}{4}})} \leq \varepsilon.$$

Let us denote by $\mathcal{R}$ the compact region of $\mathbb{R}^2$ (see Fig. 2) defined by

$$\mathcal{R} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq -A,\ x_1 + x_2 \leq R_{\frac{\varepsilon}{4}},\ x_1 - x_2 \leq R_{\frac{\varepsilon}{4}} \right\}.$$

Since the embedding $H^1(\mathcal{R}) \hookrightarrow L^2(\mathcal{R})$ is compact, we conclude from (3.2) that

$$u(\cdot + \rho(t_n), t_n) \xrightarrow[n \to +\infty]{} \tilde{u}_0 \quad \text{in} \quad L^2(x_1 > -A).$$

5) Finally, we also need to derive monotonicity properties for a part of the energy in order to get the $\dot{H}^1$ part of the convergence in (3.5). The rest of the proof (and in particular the exponential decay on the left in (3.4)) follows the lines of the paper of Martel, Merle [21].
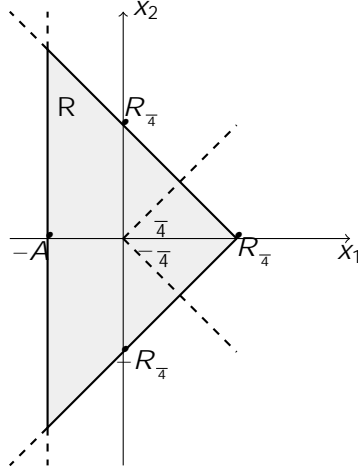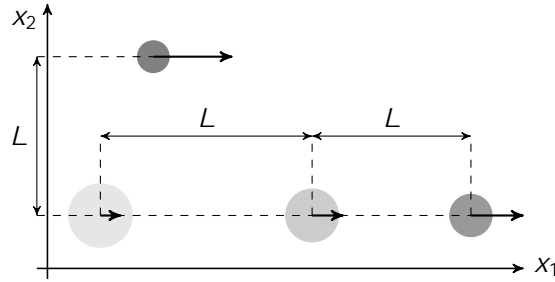
Figure 2. R is a compact set of $\mathbb{R}^2$.

Figure 3. A schematic example of admissible initial data. Solitons are represented by the disk where their mass is concentrated.

4. Stability of multi-solitons

As a consequence of the monotonicity properties, we are also able to prove the stability of the sum of N essentially non-colliding solitons.

Let $N \geq 2$ be an integer and $L > 0$. We say that

$$Q_{c_1^0}(x - z^{1,0}), \ Q_{c_2^0}(x - z^{2,0}), \ldots, Q_{c_N^0}(x - z^{N,0})$$

are N L-decoupled solitons, if

- $|z_2^{j,0} - z_2^{k,0}| \geq L$, or
- $c_k^0 > c_j^0$ and $z_1^{k,0} - z_1^{j,0} \geq L$.

See Figure 3 for an example of N L-decoupled solitons.

Theorem 4.1 (Côte, Muñoz, P., Simpson [4]. Stability of the sum of N decoupled solitons). *Assume $d = 2$. Consider a set of N solitons of the form*

$$Q_{c_1^0}(x - z^{1,0}), \ Q_{c_2^0}(x - z^{2,0}), \ldots, Q_{c_N^0}(x - z^{N,0}),$$

where each c_j^0 is a fixed positive scaling, $c_j^0 = c_k^0$ for all $j = k$, and $^{j,0} = (_1^{j,0}, _2^{j,0}) \quad \mathbb{R}^2$. Assume that the N solitons are L-decoupled.

Then there are $_0 > 0$, $C_0 > 0$ and $L_0 > 0$ depending on the previous parameters such that, for all $(0, _0)$, and for every $L > L_0$, the following holds. Suppose that $u_0 \quad H^1(\mathbb{R}^2)$ satisfies

$$(4.1) \qquad u_0 - \sum_{j=1}^{N} Q_{c_j^0}(x - ^{j,0})_{H^1} < .$$

Then there are $_1 > 0$ fixed and $^j(t) \quad \mathbb{R}^2$ defined for all $t \quad 0$ such that $u(t)$, solution of (1.1) with initial data $u(0) = u_0$ satisfies

$$(4.2) \qquad \sup_{t \ 0} u(t) - \sum_{j=1}^{N} Q_{c_j^0}(x - ^j(t))_{H^1} < C_0(+ e^{- _1 L}).$$

Note that we do not need strictly well-prepared initial data as in [22]. Instead, by using the continuity of the flow, the hypothesis on the N L-decoupled solitons ensures that after some positive time T the solitons are well-prepared, *i.e.* decoupled in the x_1 variable and arranged by increasing speed as x_1 grows.

The rest of the proof is obtained by adapting the ideas by Martel, Merle and Tsai [22] for the generalized, one dimensional KdV case.

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