Séminaire Laurent Schwartz

EDP et applications

Année 2011-2012

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Séminaire Laurent Schwartz — EDP et applications (2011-2012), Exposé n° XXXVI, 12 p.

<http://slsedp.cedram.org/item?id=SLSEDP_2011-2012____A36_0>

Séminaire Laurent-Schwartz — EDP et applications
Centre de mathématiques Laurent Schwartz, 2011-2012
Exposé n^{o} XXXVI, 1-12

On Bardina and Approximate Deconvolution Models

Roger LEWANDOWSKI *

1 Introduction

This text is based on the talk which was given at Seminar Laurent Schwartz, Ecole Polytechnique, France, April 3, 2012.

We first outline the procedure of averaging the incompressible Navier-Stokes equations when the flow is turbulent for various type of filters.

We introduce the turbulence model called Bardina's model, for which we are able to prove existence and uniqueness of a distributional solution.

In order to reconstruct some of the flow frequencies that are underestimated by Bardina's model, we next introduce the approximate deconvolution model (ADM). We prove existence and uniqueness of a "regular weak solution" to the ADM for each deconvolution order N, and then that the corresponding sequence of solutions converges to the mean Navier-Stokes Equations when N goes to infinity.

2 Filtered Navier-Stokes Equation

2.1 Navier-Stokes Equations

The incompressible Navier-Stokes Equations (NSE) that modelise the motion of a fluid are

$$\begin{cases} \partial_t \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f}, \\ \nabla \cdot \mathbf{u} &= 0, \\ \mathbf{u}(0, \mathbf{x}) &= \mathbf{u}_0(\mathbf{x}), \end{cases}$$

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where $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ is the velocity, $p = p(t, \mathbf{x})$ is the pressure rescaled by the density, assumed to be constant in the momentum equation. For turbulent flows, we know from Kolmogorov theory that one needs about $Re^{9/4}$ degrees of freedom in a unit box, which requires too much computer memory, where Re is the Reynolds number, typically larger than 10^{10} in practical situations. Turbulent models aim to compute mean fields to reduce the complexity and to stabilize numerical schemes.

2.2 General Filtering Principle by convolution

Let $(\overline{\mathbf{u}}, \overline{p})$ the mean flow field defined by

$$\overline{\mathbf{u}}(t, \mathbf{x}) = \int G_\alpha(\mathbf{x}, \mathbf{y})\mathbf{u}(\mathbf{y})d\mathbf{y}, \quad \overline{p} = \int G_\alpha(\mathbf{x}, \mathbf{y})p(\mathbf{y})d\mathbf{y},$$

for a given convolution kernel G_α. For homogeneous turbulent flows, one may take

$$G_\alpha(\mathbf{x}, \mathbf{y}) = G_\alpha(|\mathbf{x} - \mathbf{y}|), \quad \overline{\mathbf{u}} = G_\alpha \star \mathbf{u} = G_\alpha(\mathbf{u}).$$

The kernel G_α is smooth, reduces amplitude of high frequency modes and converges to the dirac function in the sense that

$$G_\alpha(\mathbf{u}) \to \mathbf{u} \quad \text{when} \quad \alpha \to 0.$$

Typical examples :

- Gaussian filter: Let $r = |\mathbf{x} - \mathbf{y}|$,

$$G_\alpha(\mathbf{r}) = \frac{1}{(4\pi)^{\frac{3}{2}}\alpha^3} e^{-r^2/\alpha^2}, \quad \alpha > 0.$$

  Notice that if $\overline{\mathbf{u}} = G_\alpha \star \mathbf{u}$,

$$\partial_\alpha \overline{\mathbf{u}} - \Delta_\mathbf{x} \overline{\mathbf{u}} = 0, \quad \overline{\mathbf{u}}|_{\alpha=0} = \mathbf{u},$$

  therefore

$$\overline{\mathbf{u}} \to \mathbf{u} \quad \text{when} \quad \alpha \to 0.$$

- Helmholz filter: G_α is the Green function of the operator $-\alpha^2\Delta + \mathrm{Id}$, $\alpha > 0$, whose fourier's transform is

$$\widehat{G}_\alpha(\mathbf{k}) = \frac{1}{1 + \alpha^2|\mathbf{k}|^2},$$

  where $\mathbf{k}$ is the wave vector, which means that $\overline{\mathbf{u}} = G_\alpha \star \mathbf{u}$ solves

$$-\alpha^2\Delta\overline{\mathbf{u}} + \overline{\mathbf{u}} = \mathbf{u}.$$

In the following, we shall consider the Helmholz filter. More generally, for any quantity ψ related to the flow,

$$-\alpha^2 \Delta\overline{\psi} + \overline{\psi} = \psi.$$

The length $\alpha > 0$ is of the same order than mean mesh size in any practical calculation, and

$$k_c = \frac{2\pi}{\alpha}$$

is the "cut-frequency" . We also denote

$$\overline{\mathbf{u}} = G\mathbf{u} \quad \mathbf{u} = A\overline{\mathbf{u}}$$

when G is invertible and (we do not write the subscript α for simplicity)

$$A = G^{-1}.$$

We aim to find approximated equations satisfied by $(\overline{\mathbf{u}}, \overline{p})$ or rather their "model substitute" $(\mathbf{w}, q)$", to describe flow scales $k \geq k_c$, $k = |\mathbf{k}|$ denotes the vawe number.

2.3 Bardina's Model

Taking the mean of the Navier-Sokes Equations yields

$$(2.1) \quad \begin{cases} \partial_t \overline{\mathbf{u}} + \nabla \cdot (\overline{\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}}) - \nu\Delta\overline{\mathbf{u}} + \nabla\overline{p} &= \overline{\mathbf{f}} + \nabla \cdot (\overline{\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}} - \overline{\mathbf{u} \otimes \mathbf{u}}), \\ \nabla \cdot \overline{\mathbf{u}} &= 0, \\ \overline{\mathbf{u}}(0, \mathbf{x}) &= \overline{\mathbf{u}_0}(\mathbf{x}), \end{cases}$$

where

$$\boldsymbol{\tau}_r = \overline{\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}} - \overline{\mathbf{u} \otimes \mathbf{u}}$$

denotes the residual stress. We assume $|\boldsymbol{\tau}_r| << 1$. For instance when $\mathbf{u}$ is C^1 and for the Helmholtz filter, it is easely checked that $||\boldsymbol{\tau}_r||_{H^1} = 0(\alpha^2)$. Neglecting the residual stress and its variations leads to consider the model for

$$(\mathbf{w}, q) \approx (\overline{\mathbf{u}}, \overline{p}),$$

which simplified the mean Navier-Stokes Equations as

$$(2.2) \quad \begin{cases} \partial_t \mathbf{w} + \nabla \cdot (\overline{\mathbf{w} \otimes \mathbf{w}}) - \nu\Delta\mathbf{w} + \nabla q &= \overline{\mathbf{f}}, \\ \nabla \cdot \mathbf{w} &= 0, \\ \mathbf{w}(0, \mathbf{x}) &= \overline{\mathbf{u}_0}(\mathbf{x}). \end{cases}$$

The boundary conditions are periodic boundary conditions. This model was introduced first in Bardina *et al* [2], and is called Bardina's model.

3 Mathematical setting of Bardina's model

3.1 Function space

Let $\mathbb{T}_3$ be the 3-D torus

$$\mathbb{T}_3 = \mathbb{R}^3/\mathcal{T}_3 \quad \text{where} \quad \mathcal{T}_3 := 2\pi\mathbb{Z}^3/L$$

for some given $L > 0$. All considered fields have a zero mean on $\mathbb{T}_3$.

$$\text{We assume} \quad \mathbf{u}_0 \in \mathbf{H}_0, \quad \mathbf{f} \in L^2([0,T]\times\mathbb{T}_3)^3, \quad \text{where}$$

$$\mathbf{H}_s = \left\{ \mathbf{w} : \mathbb{T}_3 \to \mathbb{R}^3,\ \mathbf{w} \in H^s(\mathbb{T}_3)^3, \quad \nabla\cdot\mathbf{w} = 0, \quad \int_{\mathbb{T}_3} \mathbf{w}\, d\mathbf{x} = \mathbf{0} \right\},$$

which is a closed subset of,

$$\mathbb{H}_s = \left\{ \mathbf{w} = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \widehat{\mathbf{w}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} : \quad \sum_{\mathbf{k}\in\mathcal{T}_3^\star} |\mathbf{k}|^{2s} |\widehat{\mathbf{w}}_{\mathbf{k}}|^2 < \infty,\ \mathbf{k}\cdot\widehat{\mathbf{w}}_{\mathbf{k}} = 0 \right\},$$

by noting $\forall\, \mathbf{k} = (k_1, k_2, k_3) \in \mathcal{T}_3$, $|\mathbf{k}|^2 = k_1^2 + k_2^2 + k_3^2$, and

$$(\mathbf{w}, \mathbf{v})_{\mathbf{H}_s} = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} |\mathbf{k}|^{2s} \widehat{\mathbf{w}}_{\mathbf{k}} \cdot \widehat{\mathbf{v}}_{\mathbf{k}}^\star, \quad ||\mathbf{w}||_s = (\sum_{\mathbf{k}\in\mathcal{T}_3^\star} |\mathbf{k}|^{2s} |\mathbf{w}_{\mathbf{k}}|^2)^{\frac{1}{2}}.$$

3.2 Estimate for Bardina's model with Helmholz filter

When G and $A = G^{-1} = -\alpha^2\Delta + \mathrm{I}$ is the Helmholz filter, which commutes with differential operators, Bardina's model may be written as

$$\text{(3.1)} \qquad \left\{ \begin{array}{rcl} \partial_t \mathbf{w} + A^{-1}(\nabla\cdot(\mathbf{w}\otimes\mathbf{w})) - \nu\Delta\mathbf{w} + \nabla q &=& \overline{\mathbf{f}}, \\ \nabla\cdot\mathbf{w} &=& 0, \\ \mathbf{w}(0,\mathbf{x}) &=& \overline{\mathbf{u}_0}(\mathbf{x}). \end{array} \right.$$

We add the compatibility conditions

$$\int_{\mathbb{T}_3} \mathbf{w} = \mathbf{0}, \quad \int_{\mathbb{T}_3} q = 0.$$

Take formally $A\mathbf{w} = -\alpha^2\Delta\mathbf{w} + \mathbf{w}$ as test vector field in (3.1), and integrate over $\mathbb{T}_3$. Since A is self adjoint for the L^2 inner product,

$$(A^{-1}(\nabla\cdot(\mathbf{w}\otimes\mathbf{w})), A\mathbf{w})_{L^2} = (\nabla\cdot(\mathbf{w}\otimes\mathbf{w})), \mathbf{w})_{L^2} = \int_{\mathbb{T}_3} ((\mathbf{w}\cdot\nabla)\,\mathbf{w})\cdot\mathbf{w} = 0,$$

by using $\nabla \cdot \mathbf{w} = 0$ and periodicity.

We study now the remaining terms.

- Evolution term. We note that

$$-\int_{\mathbb{T}_3} \partial_t \mathbf{w} \cdot \Delta \mathbf{w} = \int_{\mathbb{T}_3} \nabla \partial_t \mathbf{w} : \nabla \mathbf{w} = \frac{d}{2dt} \int_{\mathbb{T}_3} |\nabla \mathbf{w}|^2,$$

  therefore,

$$(\partial_t \mathbf{w}, A\mathbf{w})_{L^2} = \int_{\mathbb{T}_3} \partial_t \mathbf{w} \cdot (-\alpha^2 \Delta \mathbf{w} + \mathbf{w}) = \frac{d}{2dt} \int_{\mathbb{T}_3} \alpha^2 |\nabla \mathbf{w}|^2 + |\mathbf{w}|^2.$$

- Diffusion term. By a straighforward calculation, we get

$$(-\nu \Delta \mathbf{w}, A\mathbf{w})_{L^2} = \nu \int_{\mathbb{T}_3} |\nabla \mathbf{w}|^2 + \alpha^2 |\Delta \mathbf{w}|^2$$

- The pressure term. Since $\nabla \cdot (A\mathbf{w}) = A\,(\nabla \cdot \mathbf{w}) = 0$,

$$(\nabla q, A\mathbf{w})_{L^2} = 0.$$

In conclusion, by using usual procedures, we have for all $t > 0$ and any smooth solution $(\mathbf{w}, q)$ to (3.1),

$$\begin{aligned} &||\mathbf{w}(t,\cdot)||_0^2 + \alpha^2 ||\mathbf{w}(t,\cdot)||_1^2 + \\ &\qquad \nu \int_0^t (||\mathbf{w}(t',\cdot)||_1^2 + \alpha^2 ||\Delta \mathbf{w}(t',\cdot)||_0^2) dt' \leq \\ &||\overline{\mathbf{u}_0}(\cdot)||_0^2 + \alpha^2 ||\overline{\mathbf{u}_0}(\cdot)||_1^2 + \frac{1}{\nu} \int_0^t ||\overline{\mathbf{f}}(t',\cdot)||_{-1}^2 dt', \end{aligned} \tag{3.2}$$

which also means that under suitable assumptions about the data, that is $\mathbf{u}_0 \in \mathbf{H}_0$ and $\mathbf{f} \in L^2(\mathbb{R}, \mathbf{H}_{-1})$, $\mathbf{w}$ is bounded in,

$$L^\infty(\mathbb{R}_+, \mathbf{H}_0) \cap L^2(\mathbb{R}_+, \mathbf{H}_1),$$

with uniform bounds in α, and

$$L^\infty(\mathbb{R}_+, \mathbf{H}_1) \cap L^2(\mathbb{R}_+, \mathbf{H}_2),$$

with bounds that blow up when $\alpha \to 0$.

3.3 Existence and uniqueness of a solution to Bardina's model

Theorem 3.1. *(Layton-Lewandowski, 2006) Bardina's model (2.2) admits a unique solution* $(\mathbf{w}_\alpha, q_\alpha)$ *inthe sense of the distributions, such that*

$$\begin{aligned} &\mathbf{w}_\alpha \in L^\infty(\mathbb{R}_+, \mathbf{H}_0) \cap L^2(\mathbb{R}_+, \mathbf{H}_1), \\ &\mathbf{w}_\alpha \in L^\infty(\mathbb{R}_+, \mathbf{H}_1) \cap L^2(\mathbb{R}_+, \mathbf{H}_2), \\ &q_\alpha \in L^2(\mathbb{R}, H^1(\mathbb{T}_3)). \end{aligned}$$

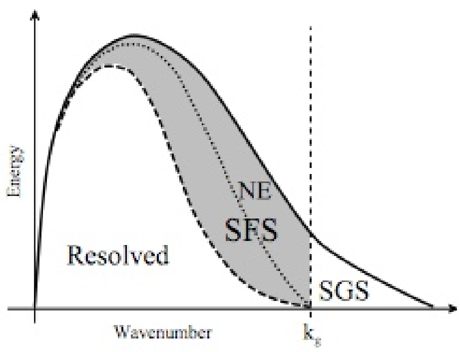
Moreover, from the sequence $(\mathbf{w}_\alpha, q_\alpha)_{\alpha>0}$*, one can extract a subsequence (still denoted by* $(\mathbf{w}_\alpha, q_\alpha)_{\alpha>0}$*) such that* $\mathbf{w}_\alpha \to \mathbf{u}$ *in* $L^2(\mathbb{R}_+ \times \mathbb{T}_3)$ *and* $q_\alpha \to p$ *in* $L^{5/3}(\mathbb{R}_+ \times \mathbb{T}_3)$ *when* $\alpha \to 0$ *and such that* $(\mathbf{u}, p)$ *is a weak dissipative solution to the Navier-Stokes Equations.*

The proof, which is detailled in [7], is based on the Galerking method, the estimate (3.2) and classical tools of functional analysis, including standard compactness lemma.

Practical calculations [3] lead to distinguish 3 types of scales:

- the resolved scales
- the subgrid scales (SGS) (filtration by the numerical scheme)
- the subfilter scales (SFS) (filtration by the convolution)

Figure 3.1: From Chow et al. [3]. ©American Meteorological Society. Reprinted with permission.



The challenge is the reduction of the SFS area.

4 Approximate Deconvolution Models (ADM)

4.1 General Setting

As said above, we aim to reduce the SFS area. To do so, we follow the procedure introduced in [10] and [11], which is based on a deconvolution operator defined by

$$D_N = \sum_{n=0}^{N} (\mathbf{I} - G)^n.$$

When G is invertible, one may expect that

$$D_N \to G^{-1} = A, \quad \text{when} \quad N \to \infty.$$

The corresponding ADM is:

$$(4.1) \qquad \begin{cases} \partial_t \mathbf{w} + \nabla \cdot (\overline{D_N(\mathbf{w}) \otimes D_N(\mathbf{w})}) - \nu \Delta \mathbf{w} + \nabla q &= \overline{\mathbf{f}}, \\ \nabla \cdot \mathbf{w} &= 0, \\ \mathbf{w}(0, \mathbf{x}) &= \overline{\mathbf{u}_0}(\mathbf{x}), \end{cases}$$

N.B. When $N = 0$, this is the Bardina's model.

The Issues are:

- Is there an alternative to the Helmholz filter,
- What kind of solution is the more suitable to the ADM,
- Can one proves the existence and uniqueness of solutions satisfying estimates that do not depend on N,
- What is the behavior of the solution when $N \to \infty$.

This will serve as a guide for the following.

Remark 4.1. *When G is the Helmholtz filter of width α, we already know that:*
- there Exists a unique distributional solution to the ADM, whose estimates depends on N, which converges to a solution to the NSE when $\alpha \to 0$ (see in [5]),
- the rate of convergence to the NSE when $\alpha \to 0$ is of order $\alpha^{1/3}$, the constant depending on N (see in [8]).

4.2 The filter and the deconvolution operator

Assume G invertible. We denote by $\widehat{G}_{\mathbf{k}}$ the symbol of the operator G, defined by the Fourier's series of the kernel still denoted by G,

$$G = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \widehat{G}_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}}, \quad A = G^{-1} = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \widehat{A}_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$\text{In addition} \quad G\mathbf{w} = G \star \mathbf{w} = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \widehat{G}_{\mathbf{k}}\ \widehat{\mathbf{w}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

The generalised Helmholz filter is defined by:

$$\forall\, \mathbf{k} \in \mathcal{T}_3^\star, \quad \widehat{G}_{\mathbf{k}} = \frac{1}{1+\alpha^{2p}|\mathbf{k}|^{2p}}, \quad \widehat{G}_\alpha = 0,$$

which is an isomorphism $\mathbb{H}_s \to \mathbb{H}_{s+2p}$, and

$$-\alpha^{2p}\Delta^p\overline{\mathbf{u}} + \overline{\mathbf{u}} = \mathbf{u}, \quad \nabla\cdot\overline{\mathbf{u}} = 0.$$

The symbol of the deconvolution operator $D_N = \sum_{0\le n\le N} (\mathbf{I}-G)^n$ is:

$$\widehat{D_{N,\mathbf{k}}} = \sum_{n=0}^{N}\left(\frac{\alpha^{2p}|\mathbf{k}|^{2p}}{1+\alpha^{2p}|\mathbf{k}|^{2p}}\right)^n = (1+\alpha^{2p}|\mathbf{k}|^{2p})\rho_{N,p,\mathbf{k}},$$

$$\rho_{N,p,\mathbf{k}} = 1 - \left(\frac{\alpha^{2p}|\mathbf{k}|^{2p}}{1+\alpha^{2p}|\mathbf{k}|^{2p}}\right)^{N+1}.$$

The main properties of this operator are:

- $1 \le \widehat{D_{N,\mathbf{k}}} \le N+1, \quad \forall\, \mathbf{k}\in\mathcal{T}_3,$
- $\widehat{D_{N,\mathbf{k}}} \approx (N+1)\dfrac{1+\alpha^{2p}|\mathbf{k}|^{2p}}{\alpha^{2p}|\mathbf{k}|^{2p}}$ for large $|\mathbf{k}|$, $\quad \lim_{|\mathbf{k}|\to+\infty} \widehat{D_{N,\mathbf{k}}} = N+1,$
- $\widehat{D_{N,\mathbf{k}}} \le (1+\alpha^{2p}|\mathbf{k}|^{2p}) = \widehat{A}_{\mathbf{k}}, \quad \forall\, \mathbf{k}\in\mathcal{T}_3,$
- $\forall\, \mathbf{k}\in\mathcal{T}_3$ fixed $\quad \widehat{D}_{N,\mathbf{k}} \to 1+\alpha^{2p}|\mathbf{k}|^{2p} = \widehat{A}_{\mathbf{k}}, \quad$ as $N\to+\infty,$
- $\forall\, \mathbf{v} \in L^2([0,T],\mathbf{H}_{2p}),\ D_N(\mathbf{v}) \to A_p\mathbf{v}$ in $L^2([0,T]\times\mathbb{T}_3)^3$, as $N\to\infty$.

4.3 Notion of Solution and existence result

Definition 4.1 (Regular Weak solution)**.** *We say that the couple* $(\mathbf{w}, q)$ *is a "regular weak solution" to the ADM (4.1) if and only if the three following items are satisfied:*

1) REGULARITY

$$\mathbf{w} \in L^2([0,T]; \mathbf{H}_{1+p}) \cap C([0,T]; \mathbf{H}_p), \tag{4.2}$$

$$\partial_t \mathbf{w} \in L^2([0,T]; \mathbf{H}_0) \tag{4.3}$$

$$q \in L^2([0,T]; H^1(\mathbb{T}_3)), \tag{4.4}$$

2) INITIAL DATA

$$\lim_{t \to 0} \|\mathbf{w}(t,\cdot) - \overline{\mathbf{u}_0}\|_{\mathbf{H}_p} = 0, \tag{4.5}$$

3) WEAK FORMULATION

$$\forall \, \mathbf{v} \in L^2([0,T]; H^1(\mathbb{T}_3)^3), \tag{4.6}$$

$$\begin{cases} \displaystyle\int_0^T \int_{\mathbb{T}_3} \partial_t \mathbf{w} \cdot \mathbf{v} - \int_0^T \int_{\mathbb{T}_3} \overline{D_N(\mathbf{w}) \otimes D_N(\mathbf{w})} : \nabla \mathbf{v} + \\ \displaystyle\nu \int_0^T \int_{\mathbb{T}_3} \nabla \mathbf{w} : \nabla \mathbf{v} \\ \displaystyle + \int_0^T \int_{\mathbb{T}_3} \nabla q \cdot \mathbf{v} = \int_0^T \int_{\mathbb{T}_3} \overline{\mathbf{f}} \cdot \mathbf{v}. \end{cases} \tag{4.7}$$

Theorem 4.1. *(Berselli-Lewandowski 2012) Assume* $p > 3/4$*. The ADM has a unique regular weak solution. Moreover, when* $p \geq 1$*,*

$$\partial_t \mathbf{w} \in L^2([0,T], \mathbf{H}_{p-1}), \quad q \in L^2([0,T], H^p(\mathbb{T}_3)).$$

The proof of this theorem is detailled in [4]. Apart the usual tools of functional analysis, the result is based on the following estimates. We formally take $AD_N\mathbf{w}$ as test in the ADM, to get the following energy equality:

$$\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w})\|^2 + \|\nabla A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w})\|^2 = (A^{\frac{1}{2}} D_N^{\frac{1}{2}} \mathbf{f}, A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}))$$

The basic properties of $\widehat{D}_{N,\mathbf{k}}$ combined with the energy estimate satisfied by $A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w})$, yield the following formal estimates:

Label	Variable	bound	order
$a)$	$A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w})$	$L^\infty([0,T];\mathbf{H}_0) \cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
$b)$	$D_N^{1/2}(\mathbf{w})$	$L^\infty([0,T];\mathbf{H}_0) \cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
$c)$	$D_N^{1/2}(\mathbf{w})$	$L^\infty([0,T];\mathbf{H}_p) \cap L^2([0,T];\mathbf{H}_{1+p})$	$O(\alpha^{-p})$
$d)$	$\mathbf{w}$	$L^\infty([0,T];\mathbf{H}_0) \cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
$e)$	$\mathbf{w}$	$L^\infty([0,T];\mathbf{H}_p) \cap L^2([0,T];\mathbf{H}_{1+p})$	$O(\alpha^{-p})$
$f)$	$D_N(\mathbf{w})$	$L^\infty([0,T];\mathbf{H}_0) \cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
$g)$	$D_N(\mathbf{w})$	$L^\infty([0,T];\mathbf{H}_p) \cap L^2([0,T];\mathbf{H}_{p+1})$	$O(\alpha^{-p} \cdot (N+1)^{1/2})$
$h)$	$\partial_t \mathbf{w}$	$L^2([0,T];\mathbf{H}_0)$, for $p > \frac{3}{4}$	$O(\alpha^{-p})$

4.4 Asymptotic behavior

The interesting problem is the problem of letting N running to ∞. We have the following result.

Theorem 4.2. *(Berselli-Lewandowski 2012) Let $(\mathbf{w}_N, q_N)$ denote the solution to the ADM at N. From the sequence $(\mathbf{w}_N, q_N)_{N\in\mathbb{N}}$ one can extract a sub-sequence (still denoted $(\mathbf{w}_N, q_N)_{N\in\mathbb{N}}$) such that*

$$\begin{array}{ll} \mathbf{w}_N \to \mathbf{w} & \begin{array}{l} \textit{weakly in } L^2([0,T];\mathbf{H}_{1+p}) \cap L^\infty([0,T];\mathbf{H}_p) \\ \textit{strongly in } L^r([0,T];\mathbf{H}_p(\mathbb{T}_3)^3),\ \forall\, 1 \le r < +\infty, \end{array} \\ q_N \to q & \textit{weakly in } L^2([0,T];H^1(\mathbb{T}_3) \cap L^{5/3}([0,T];W^{2p,5/3}(\mathbb{T}_3)) \end{array}$$

and such that the system

$$\left\{ \begin{array}{rcl} \partial_t \mathbf{w} + \nabla \cdot (\overline{A\mathbf{w} \otimes A\mathbf{w}}) - \nu \Delta \mathbf{w} + \nabla q &=& \overline{\mathbf{f}}, \\ \nabla \cdot \mathbf{w} &=& 0, \\ \mathbf{w}(0,\mathbf{x}) &=& \overline{\mathbf{u}_0}(\mathbf{x}), \end{array} \right.$$

holds in the sense of the distributions.

We call the limit system the mean Navier-Stokes Equations. The following explain the motivation of this terminology. Let us consider

$$(\mathbf{u}, p) = (A\mathbf{w}, Aq)$$

which can also be written as

$$(\mathbf{w}, q) = (G\mathbf{u}, Gp) = (\overline{\mathbf{u}}, \overline{p}).$$

The mean NSE becomes

$$\begin{cases} \partial_t \overline{\mathbf{u}} + \nabla \cdot (\overline{\mathbf{u} \otimes \mathbf{u}}) - \nu \Delta \overline{\mathbf{u}} + \nabla \overline{p} & = & \overline{\mathbf{f}} \\ & = & \overline{\partial_t \mathbf{u} + \nabla \cdot \mathbf{u} \otimes \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p}, \\ \nabla \cdot \overline{\mathbf{u}} & = & 0 = \overline{\nabla \cdot \mathbf{u}}, \\ \overline{\mathbf{u}}(0, \mathbf{x}) & = & \overline{\mathbf{u}_0}(\mathbf{x}), \end{cases}$$

which is the NSE that have been averaged, since the operator G commutes with differential operators, hence the name "mean NSE" (MNSE), precisely equations (2.1). In fact, $(\mathbf{u}, p)$ is a "very weak" solution to the NSE.

Remark 4.1. *The solution to the MNSE is dissipative, which means that it satisfies an energy inequality.*
The critical exponent $3/4$ *is not optimal: existence and uiniqueness can be obtained up to* $p > 1/2$*, which is an improvement by H. Ali [1].*

The proof is detailled in [4]. The main steps consists first in obtaining additional shars estimates, set in the array below.

Variable	bound	order
$\mathbf{w}_N$	$L^\infty([0,T];\mathbf{H}_0) \cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
$\mathbf{w}_N$	$L^\infty([0,T];\mathbf{H}_p) \cap L^2([0,T];\mathbf{H}_{1+p})$	$O(\alpha^{-1})$
$D_N(\mathbf{w}_N)$	$L^\infty([0,T];\mathbf{H}_0) \cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
$\partial_t \mathbf{w}_N$	$L^2([0,T] \times \mathbb{T}_3)^3$	$O(\alpha^{-1})$
q_N	$L^2([0,T];H^1(\mathbb{T}_3)) \cap L^{5/3}([0,T];W^{2,5/3}(\mathbb{T}_3))$	$O(\alpha^{-1})$
$\partial_t D_N(\mathbf{w}_N)$	$L^{4/3}([0,T];\mathbf{H}_{-1})$	$O(1)$

Next we show that:

- $AD_N(\mathbf{V}_n) \subset \mathbf{V}_n$, where $\mathbf{V}_n$ is a finite dimensional Galerkin space,
- we justify that $AD_N\mathbf{w}$ is a possible test, when $3/4 < p \leq 1$ first, $1 < p$ next
- we make use of the main properties of $\widehat{D}_{N,\mathbf{k}}$, which yields in particular $||\mathbf{w}||_s \leq ||D_N\mathbf{w}||_s \leq ||A^{1/2}D_N^{1/2}\mathbf{w}||_s$, $\forall\, s$.
- we prove a compactness property satisfied by $(D_N\mathbf{w}_N)_{N\in\mathbb{N}}$ thanks to Aubin-Lions Lemma, and use it to pass to the limit in the term $D_N\mathbf{w}_N \otimes D_N\mathbf{w}_N$, which is the main task.

References

[1] H. Ali, *Mathematical study of some turbulence models.* PhD Thesis of the University of Rennes 1, 2011.

[2] J. Bardina, J.H. Ferziger and W.C. Reynold. *Improved turbulence models based on large eddy simulation of homogeneous, invompressible, turbulent ows.* Technical Report No.TF-19, Department of mechanical Engineering, Stanford University, Stanford, 1983.

[3] F.K. Chow, R.L. Street, M. Xue and J.H. Ferziger. Explicit filtering and reconstruction turbulence modeling for large-eddy simulation of neutral boundary layer ow, *Journal of the Atmospheric Sciences,* 62(7): 2058-2077, 2005.

[4] L. Berselli and R. Lewandowski. Convergence of Approximate Deconvolution Models to the mean Navier-Stokes equations *Annales de l'Institut Henri Poincare (C), Non Linear Analysis*, 29:171-198, 2012.

[5] A. Dunca and Y. Epshteyn. On the Stolz-Adams deconvolution model for the large-eddy simulation of turbulent flows. *SIAM J. Math. Anal.*, 37(6):1890–1902 (electronic), 2006.

[6] W. Layton and R. Lewandowski. A simple, accurate and stable scale similarity model for LES: Energy balance and existence of weak solutions. *Appl. Math. Lett.*, 4:1205–1209, 2003.

[7] W. Layton and R. Lewandowski. On a well posed turbulence model. *Discrete and Continuous Dynamical Systems series B*, 6(1):111-128, 2006.

[8] W. J. Layton and R. Lewandowski. Residual stress of approximate deconvolution models of turbulence. *J. Turbul., 7:Paper 46, 21 pp. (electronic)*, 2006.

[9] W. Layton and R. Lewandowski, A high accuracy Leray-deconvolution model of turbulence and its limiting behavior *Analysis and Applications* 6(1):23-49, 2008.

[10] S. Stoltz and N. A. Adams. An approximate deconvolution procedure for large-eddy simulation. *Phys. Fluids*, 11:1699–1701, 1999.

[11] S. Stolz, N. A. Adams, and L. Kleiser. An approximate deconvolution model for large-eddy simulation with application to incompressible wall-bounded flows. *Phys. Fluids*, 13(4):997–1015, 2001.