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## ELLIPTIC PROBLEMS WITH INTEGRAL DIFFUSION

YANNICK SIRE

ABSTRACT. In this paper, we review several recent results dealing with elliptic equations with non local diffusion. More precisely, we investigate several problems involving the fractional laplacian. Finally, we present a conformally covariant operator and the associated singular and regular Yamabe problem.

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### 1. FRACTIONAL POWERS OF ELLIPTIC OPERATORS

For an operator  $L$ , self-adjoint, non negative, one can define its fractional powers by means of spectral theory, namely

$$\forall s \in (0, 1), L^s f = \frac{1}{\Gamma(1-s)} \int_0^\infty t^{-s} L e^{-Lt} f dt.$$

or

$$\forall s \in (0, 1), L^s f = \frac{\sin(\pi(1-s))}{\pi} \int_0^\infty \lambda^{s-1} L (L + \lambda \text{Id})^{-1} f d\lambda.$$

We now give two examples. In  $\mathbb{R}^n$ , we consider the Fourier multiplier of symbol  $|\xi|^{2s}$  which is the fractional laplacian, denoted  $(-\Delta)^s$  for  $s \in (0, 1)$ . In a bounded open (Lipschitz) set  $\Omega$ , we define powers of the Dirichlet laplacian by means of spectral theory. More precisely, let  $\{\varphi_k\}_{k=1}^\infty$  denote an orthonormal basis of  $L^2(\Omega)$  consisting of eigenfunctions of  $-\Delta$  in  $\Omega$  with

homogeneous Dirichlet boundary conditions, associated to the eigenvalues  $\{\mu_k\}_{k=1}^\infty$ , i.e.

$$\begin{cases} -\Delta\varphi_k = \mu_k\varphi_k & \text{in } \Omega \\ \varphi_k = 0 & \text{on } \partial\Omega. \end{cases}$$

For any  $u \in C_c^\infty(\Omega)$ , we define

$$(-\Delta)^s u = \sum_{k=1}^\infty \mu_k^s u_k \varphi_k,$$

where

$$u = \sum_{k=1}^\infty u_k \varphi_k, \quad \text{and} \quad u_k = \int_\Omega u \varphi_k \, dx.$$

One can extend by density the previous definition to the Hilbert space

$$H = \{u \in L^2(\Omega) : \|u\|_H^2 = \sum_{k=1}^\infty \mu_k^s |u_k|^2 < +\infty\}.$$

The operator  $(-\Delta)^s$  in  $\mathbb{R}^n$  is non local. For this reason, it might be difficult to analyse it. Fortunately, Caffarelli and Silvestre (see [CS07]) proved the following fact: let  $s \in (0, 1)$  and  $v \in H^1(x^{1-2s})$  given by

$$v := \arg \min \left\{ \int_{\mathbb{R}_+ \times \mathbb{R}^n} x^{1-2s} |\nabla w|^2 \, dx \, dy : w|_{\mathbb{R}^n \times \{0\}} = u \right\}$$

Then  $v$  solves

$$\begin{cases} \operatorname{div}(x^{1-2s} \nabla v) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^n \\ v = u & \text{on } \mathbb{R}^n. \end{cases}$$

and one has for the Diriclet-to-Neumann map

$$\Gamma_s : \begin{cases} H^s(\mathbb{R}^n) \rightarrow H^{-s}(\mathbb{R}^n) \\ v \mapsto \Gamma_s(v) = f := -x^{1-2s} u_x|_{\mathbb{R}^n} = (-\Delta)^s v \end{cases}$$

up to some multiplicative constant.

## 2. SYMMETRY AND LIOUVILLE RESULTS FOR FRACTIONAL NON LOCAL EQUATIONS

A well-known conjecture by De Giorgi was stated as follows thirty years ago: consider a smooth bounded solution of

$$-\Delta u = u - u^3 \quad \text{in } \mathbb{R}^n$$

such that  $\partial_n u > 0$ . Then all the level sets of  $u$ , i.e.  $\{u = c\}$ , are hyperplanes for  $n \leq 8$  or equivalently: there exists a function  $u_0$  and  $\omega \in \mathbb{S}^{n-1}$  such that

$$u(x) = u_0(\omega \cdot x).$$

The conjecture was solved by Ghoussoub and Gui [GG98] for  $n = 2$ , Ambrosio and Cabré [AC00] (and Alberti for general nonlinearities [AAC01]) for  $n = 3$ . For  $4 \leq n \leq 8$ , one major progress has been performed by Savin

[Sav09] with an additional (natural) assumption. Finally, there exists a non flat bounded smooth solution for  $n \geq 9$  and it has been constructed by Del Pino, Kowalczyk and Wei in [dKW11].

The goal is now to see what remains of the De Giorgi conjecture symmetry result if we change the laplace operator in the fractional laplacian and the ambient space  $\mathbb{R}^n$  into a manifold.

**2.1. Symmetry results.** We prove the following theorem in [SV09a] and [CS10].

**Theorem 2.1.** *Let  $s \in (0, 1)$ . Let  $u$  be a bounded smooth solution of ( $f$  is  $C^1$ )*

$$(1) \quad (-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^2.$$

such that

$$\partial_{x_2} u > 0.$$

Then there exists  $\omega \in \mathbb{S}^1$  and  $u_o : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$u(x) = u_o(\omega \cdot x).$$

The previous theorem holds in two dimensions. For  $n = 3$ , sharp energy inequalities and symmetry result for  $s \geq 1/2$  have been obtained by Cabré et Cinti (see [CC10]). For  $n \geq 4$ , the problem is completely open. One crucial question is the critical dimension in connection with a non local version of De Giorgi conjecture. The results in [SV10a] allow to think that the critical dimension for  $s \geq \frac{1}{2}$  is 8, as in the standard De Giorgi conjecture. For  $s < \frac{1}{2}$ , this is completely unsettled and would require to develop Bernstein type theorems for fractional minimal surfaces (see [CRS10]).

We now provide a sketch of the proof of Theorem 2.1 as written in [SV09a].

We denote by  $P_s$  the Poisson kernel of the operator  $\operatorname{div}(x^{1-2s}\nabla)$ . The extension  $u = P_s * v$  solves

$$(2) \quad \begin{cases} \operatorname{div}(x^{1-2s}\nabla u) = 0 & \text{in } \mathbb{R}_+^{n+1} \\ \partial_\nu u = f(u) & \text{on } \partial\mathbb{R}_+^{n+1}. \end{cases}$$

The monotonicity assumption in the theorem reads as stability in one more dimension (see [SV09b]), i.e.

$$\int_{\mathbb{R}_+^{n+1}} x^{1-2s} |\nabla \xi|^2 - \int_{\mathbb{R}^n} f'(u) \xi^2 \geq 0$$

for any  $\xi \in C_0^\infty(\mathbb{R}_+^{n+1})$ . We plug  $|\nabla_y u| \psi$  into the stability to get the geometric following Poincaré inequality:

$$\int_{\mathcal{R}_+^{n+1}} x^{1-2s} \psi^2 \left( \mathcal{K}^2 |\nabla_y u|^2 + |\nabla_L |\nabla_y u||^2 \right) \leq \int_{\mathbb{R}_+^{n+1}} x^{1-2s} |\nabla_y u|^2 |\nabla \psi|^2$$

where  $\mathcal{K}$  is the length of the second fundamental form of a level set of  $u$ . We then take  $\psi$  to be a capacity function and we use energy estimates (holding only in  $\mathbb{R}^2$ ) to make  $\mathcal{K} = 0$ , hence the result.

In [CS10], we provide an alternative proof based on a characterization of the stability and a Liouville theorem à la Bérestycki-Caffarelli-Nirenberg.

As previously proved, everything boils down to a one-dimensional profile. The following existence theorem can be found in [CS10]

**Theorem 2.2.** *Let  $f \in C^{1,\beta}$  with  $0 < \beta < 1$ . Then there exists a bounded solution connecting  $-1$  to  $1$ , nondecreasing of*

$$(-\partial_{xx})^s u = f(u)$$

*if and only if  $G' = -f$*

$$G'(-1) = G'(1) = 0 \quad \text{and} \quad G > G(-1) = G(1) \quad \text{in} \quad (-1, 1).$$

**2.2. Liouville result.** In order to understand better what is the influence of the power  $s \in (0, 1)$ , we have the following theorem (see [DS10]).

**Theorem 2.3.** *Let  $\beta \in (0, 1)$  and  $f \in C^{1,\beta}(\mathbb{R})$  function such that  $f \geq 0$ . Let  $v \in C^2(\mathbb{R}^n)$  be a stable bounded solution of*

$$(-\Delta)^s v = f(v).$$

*Then*

- *If  $s \in [\frac{1}{2}, 1]$ ,  $v$  is constant for  $n \leq 3$ .*
- *If  $s \in (0, \frac{1}{2})$ ,  $v$  is constant for  $n \leq 2$ .*

We however do not know if the dimensions in the previous theorem are sharp.

**2.3. The case of manifolds.** We now turn to fractional Allen-Cahn equations on Riemannian manifolds. Let  $(M, g)$  be a complete, connected, smooth Riemannian manifold without boundary. We denote  $\Delta_g$  the Laplace-Beltrami operator on  $M$ . We want to study the properties of special solutions of  $s \in (0, 1)$

$$(-\Delta_g)^s u = f(u) \quad \text{on } M.$$

We will be considering stable solutions, i.e. for any  $\xi$  smooth on  $M$

$$\int_M |(-\Delta_g)^{s/2} \xi|^2 - f'(u) \xi^2 \geq 0.$$

We prove the following theorems (see [SV10b])

**Theorem 2.4.** *Let  $(M, g)$  be a compact manifold and  $v : M \rightarrow \mathbb{R}$  be a smooth stable bounded solution of*

$$(3) \quad (-\Delta_g)^{1/2} v = f(v),$$

*Assume furthermore that*

$$Ric_g \geq 0$$

*and  $Ric_g$  does not vanish identically.*

Then  $v$  is constant.

**Theorem 2.5.** *Assume that the metric on  $\mathcal{M} = M \times \mathbb{R}^+$  is given by  $\bar{g} = g + |dx|^2$ , that  $M$  is complete, and*

$$\text{Ric}_g \geq 0,$$

with  $\text{Ric}_g$  not vanishing identically.

Assume also that, for any  $R > 0$ , the volume of the geodesic ball  $B_R$  in  $M$  (measured with respect to the volume element  $dV_g$ ) is bounded by  $C(R+1)^2$ , for some  $C > 0$ .

Then every bounded stable weak solution  $u$  is constant.

**Theorem 2.6.** *Assume that the metric on  $\mathcal{M} = M \times \mathbb{R}^+$  is given by  $\bar{g} = g + |dx|^2$  and  $\text{Ric}_g$  vanishes identically.*

Assume also that, for any  $R > 0$ , the volume of the geodesic ball  $B_R$  in  $M$  (measured with respect to the volume element  $dV_g$ ) is bounded by  $C(R+1)^2$ , for some  $C > 0$ .

Then for every  $x > 0$  and  $c \in \mathbb{R}$ , every connected component of the submanifold

$$\mathcal{S}_x = \{y \in M, u(x, y) = c\}$$

is a geodesic.

We refer to [BM09] for results in negatively curved framework (hyperbolic space). For the case  $s = 1$  (the standard laplacian), we refer the reader to [FSV11].

### 3. REGULARITY OF RADIAL EXTREMAL SOLUTIONS

**3.1. Introduction.** Consider

$$(4) \quad \begin{cases} (-\Delta)^s u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

- $\Omega$ : smooth bounded set of  $\mathbb{R}^n$ ,  $n \geq 2$
- $f$  is smooth, nondecreasing such that

$$(5) \quad f(0) > 0, \quad \text{and} \quad \lim_{u \rightarrow +\infty} \frac{f(u)}{u} = +\infty.$$

We define in the following way weak solutions for equation (4).

**Definition 3.1.** *A measurable function  $u$  in  $\Omega$  such that  $\int_{\Omega} |u| \varphi_1 dx < +\infty$  and  $\int_{\Omega} f(u) \varphi_1 dx < +\infty$ , is a weak solution if*

$$(6) \quad \int_{\Omega} u \psi dx = \lambda \int_{\Omega} f(u) (-\Delta)^{-s} \psi dx,$$

for all  $\psi \in C_c^\infty(\Omega)$  and where  $\varphi_1$  is the first eigenfunction of  $-\Delta$  with homogeneous boundary conditions.

**3.2. Existence of solutions.** We prove the following theorem see [CDDS11].

**Theorem 3.2.** *Let  $s \in (0, 1)$ . There exists  $\lambda^* > 0$  such that*

- *for  $0 < \lambda < \lambda^*$ , there exists a minimal solution  $u_\lambda \in H \cap L^\infty(\Omega)$ . In addition,  $u_\lambda$  is semi-stable and increasing with  $\lambda$ .*
- *for  $\lambda = \lambda^*$ , the function  $u^* = \lim_{\lambda \nearrow \lambda^*} u_\lambda$  is a weak solution. We call  $\lambda^*$  the extremal value of the parameter and  $u^*$  the extremal solution.*
- *for  $\lambda > \lambda^*$ , there is no solution  $u \in H \cap L^\infty(\Omega)$ .*

It is proved in [CDDS11] that bounded solutions to the equation are smooth. The remaining question is the regularity of the extremal solution  $u^*$ , hence its  $L^\infty$  bound.

**3.3. Regularity of extremal solutions.** The following theorem provides an answer (see [CDDS11])

**Theorem 3.3.** *Assume  $n \geq 2$  and let  $u^*$  be the extremal solution when  $\Omega = B_1$  and  $u^*$  is radial. We have that:*

- *(a) If  $n < 2(s + 2 + \sqrt{2(s + 1)})$  then  $u^* \in L^\infty(B_1)$ .*
- *(b) If  $n \geq 2(s + 2 + \sqrt{2(s + 1)})$ , then for any  $\mu < n/2 - 1 - \sqrt{n - 1} - s$ , there exists a constant  $C > 0$  such that  $u^*(x) \leq C|x|^{-\mu}$  for all  $x \in B_1$ .*

- (1) We do not know if the bound  $n < 2(s + 2 + \sqrt{2(s + 1)})$  is optimal for the regularity of  $u^*$ . We note however that  $\lim_{s \rightarrow 1^-} 2(s + 2 + \sqrt{2(s + 1)}) = 10$ , and that the extremal solution of

$$(7) \quad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

is singular when  $\Omega = B_1$ ,  $f(u) = e^u$ , and  $n = 10$  (see [JL73]).

- (2) In particular, for any  $2 \leq n \leq 6$ , any  $s \in (0, 1)$ , and any smooth nondecreasing  $f$  with the suitable assumptions, the extremal solution is always bounded, hence smooth.

#### 4. CONFORMAL GEOMETRY AND FRACTIONAL COVARIANT OPERATORS

This section is devoted to the study of conformally covariant operators of fractional type. We will focus on singular fractional Yamabe problem. We first recall what is the singular Yamabe problem.

Let  $(M^n, \bar{g})$  be a compact Riemannian manifold,  $n \geq 3$ . If  $\Lambda \subset M$  is any closed set, then the ‘standard’ singular Yamabe problem concerns the existence and geometric properties of complete metrics  $g = u^{\frac{4}{n-2}} \bar{g}$  with constant scalar curvature, i.e.

$$(8) \quad \Delta_{\bar{g}} u + \frac{n-2}{4(n-1)} R^{\bar{g}} u = R^g u^{\frac{n+2}{n-2}}, \quad u > 0, \quad \lim_{x \rightarrow \Lambda} u = +\infty$$

Several well-known results deal with the singular Yamabe problem. If  $R^g < 0$  we have existence quite generally if  $\Lambda$  is sufficiently large in a capacity sense (see [Lab03]). If  $R^g > 0$  existence is only known when  $\Lambda$  is a smooth submanifold (possibly with boundary) of dimension  $k < (n - 2)/2$  (see [MP96]).

A geometric counterpart of the previous results is the following theorem by Schoen and Yau (see [SY88]).

**Theorem 4.1.** *If  $(M, h)$  is a compact manifold with locally conformally flat metric  $h$  with positive scalar curvature, then the developing map  $D$  from the universal cover  $\widetilde{M}$  to  $\mathbb{S}^n$  is injective, and moreover,  $\Lambda := \mathbb{S}^n \setminus D(\widetilde{M})$  has Hausdorff dimension less than or equal to  $(n - 2)/2$ . More generally, they also showed that if  $\Omega = \mathbb{S}^n \setminus \Lambda$  carries a complete metric  $g = u^{\frac{4}{n-2}} \bar{g}$  (where  $\bar{g}$  is the standard round metric) with positive scalar curvature and bounded Ricci curvature, then  $\dim \Lambda \leq (n - 2)/2$ .*

**4.1. Construction of conformally covariant operators.** The conformal Laplacian, which is the operator appearing as the linear part of (8), fits into a holomorphic family of conformally covariant elliptic pseudodifferential operators. The operators in this family of positive even integer order are the GJMS operators, and these have a central role in conformal geometry. We first define the family of fractional conformal powers of the Laplacian. As we have already indicated, the linear operator which appears as the first two terms on the left in (8) is known as the conformal Laplacian associated to the metric  $\bar{g}$ , and denoted  $P_1^{\bar{g}}$ . It is conformally covariant in the sense that if  $f$  is any (smooth) function and  $g = u^{\frac{4}{n-2}} \bar{g}$  for some  $u > 0$ , then

$$(9) \quad P_1^{\bar{g}}(uf) = u^{\frac{n+2}{n-2}} P_1^g(f).$$

Setting  $f \equiv 1$  in (9) yields the familiar relationship (8) between the scalar curvatures  $R^{\bar{g}}$  and  $R^g$ .  $P_1$  is the first in a sequence of conformally covariant elliptic operators,  $P_k$ , which exist for all  $k \in \mathbb{N}$  if  $n$  is odd, but only for  $k \in \{1, \dots, n/2\}$  if  $k$  is even. The first construction of these operators, by Graham-Jenne-Mason-Sparling [GJMS92] (for which reason they are known as the GJMS operators), proceeded by trying to find lower order geometric correction terms to  $\Delta^k$  in order to obtain nice transformation properties under conformal changes of metric. Beyond the case  $k = 1$  which we have already discussed, the operator

$$P_2 = \Delta^2 + \delta a_n Rg + b_n Ric d + \frac{n-4}{2} Q_2,$$

called the Paneitz operator (here  $Q_2$  is the standard  $Q$ -curvature), had also been discovered much earlier than the operators  $P_k$  with  $k > 2$ .

This leads naturally to the question whether there exist any conformally covariant pseudodifferential operators of noninteger order. The breakthrough result, by Graham and Zworski [GZ03], was that if  $(M, [\bar{g}])$  is a smooth compact manifold endowed with a conformal structure, then the operators  $P_k$  can be realized as residues at the values  $\gamma = k$  of the meromorphic

family  $S(n/2 + \gamma)$  of scattering operators associated to the Laplacian on any Poincaré-Einstein manifold  $(X, G)$  for which  $(M, [\bar{g}])$  is the conformal infinity. These are the ‘trivial’ poles of the scattering operator, so-called because their location is independent of the interior geometry; One obtains a holomorphic family of elliptic pseudodifferential operators  $P_\gamma^{\bar{g}}$  (which patently depends on the filling  $(X, G)$ ). An alternate construction of these operators has been obtained by Juhl, and his monograph [Juh09] describes an intriguing general framework for studying conformally covariant operators, see also [Juh].

For various technical reasons, we focus here only on the operators  $P_\gamma$  when  $\gamma \in \mathbb{R}$ ,  $|\gamma| \leq n/2$ . These have the following properties: first,  $P_0 = \text{Id}$ , and more generally,  $P_k$  is the  $k^{\text{th}}$  GJMS operator,  $k = 1, \dots, n/2$ ; next,  $P_\gamma$  is a classical elliptic pseudodifferential operator of order  $2\gamma$  with principal symbol  $\sigma_{2\gamma}(P_\gamma^{\bar{g}}) = |\xi|_{\bar{g}}^{2\gamma}$ , hence (since  $M$  is compact),  $P_\gamma$  is Fredholm on  $L^2$  when  $\gamma > 0$ ; if  $P_\gamma$  is invertible, then  $P_{-\gamma} = P_\gamma^{-1}$ ; finally,

$$(10) \quad \text{if } g = u^{\frac{4}{n-2\gamma}} \bar{g}, \quad \text{then } P_\gamma^{\bar{g}}(uf) = u^{\frac{n+2\gamma}{n-2\gamma}} P_\gamma^g(f)$$

for any smooth function  $f$ . Generalizing the formulæ for scalar curvature ( $\gamma = 1$ ) and the Paneitz-Branson  $Q$ -curvature ( $\gamma = 2$ ), we make the definition that for any  $0 < \gamma \leq n/2$ ,  $Q_\gamma^{\bar{g}}$ , the  $Q$ -curvature of order  $\gamma$  associated to a metric  $\bar{g}$ , is given by

$$(11) \quad Q_\gamma^{\bar{g}} = P_\gamma^{\bar{g}}(1).$$

Generalizing (8), consider the ‘fractional Yamabe problem’: given a metric  $\bar{g}$  on a compact manifold  $M$ , find  $u > 0$  so that if  $g = u^{4/(n-2\gamma)} \bar{g}$ , then  $Q_\gamma^g$  is constant. This amounts to solving

$$(12) \quad P_\gamma^{\bar{g}} u = Q_\gamma^g u^{\frac{n+2\gamma}{n-2\gamma}}, \quad u > 0,$$

for  $Q_\gamma^g = \text{const}$ . More generally, we can simply seek metrics  $g$  which are conformally related to  $\bar{g}$  and such that  $Q_\gamma^g \geq 0$  or  $Q_\gamma^g < 0$  everywhere.

This fractional Yamabe problem has now been solved in many cases where the positive mass theorem is not needed [QG10], and further work on this is in progress.

As described earlier, it is also interesting to construct complete metrics of constant (positive)  $Q_\gamma$  curvature on open subdomains  $\Omega = M \setminus \Lambda$ , or in other words, to find metrics  $g = u^{4/(n-2\gamma)} \bar{g}$  which are complete on  $\Omega$  and such that  $u$  satisfies (12) with  $Q_\gamma^g$  a constant. This is the fractional singular Yamabe problem. In the first few integer cases it is known that the positivity of the curvature places restrictions on  $\dim \Lambda$  (see [SY88], [MP96] for the case  $\gamma = 1$ , [CHY04] for  $\gamma = 2$ , and [Gon05] for the analogous problem for the closely related  $\sigma_k$  curvature).

**4.2. Results.** The following results can be found in [dMMS10].

**Theorem 4.2.** *Suppose that  $(M^n, \bar{g})$  is compact and  $g = u^{\frac{4}{n-2\gamma}} \bar{g}$  is a complete metric on  $\Omega = M \setminus \Lambda$ , where  $\Lambda$  is a smooth  $k$ -dimensional submanifold and  $u$  is polyhomogeneous along  $\Lambda$  with leading exponent  $-n/2 + \gamma$ . If  $0 < \gamma \leq \frac{n}{2}$  and  $Q_\gamma^g > 0$  everywhere for any choice of Poincaré-Einstein fillings, then  $n$ ,  $k$  and  $\gamma$  are restricted by the inequality*

$$(13) \quad \Gamma\left(\frac{n}{4} - \frac{k}{2} + \frac{\gamma}{2}\right) / \Gamma\left(\frac{n}{4} - \frac{k}{2} - \frac{\gamma}{2}\right) > 0.$$

Remark:  $k < (n-2\gamma)/2 \rightarrow \Gamma(\frac{n}{4} - \frac{k}{2} + \frac{\gamma}{2}) / \Gamma(\frac{n}{4} - \frac{k}{2} - \frac{\gamma}{2}) > 0$  and equivalence for  $\gamma = 1$ .

**Theorem 4.3.** *Let  $\Gamma$  be a discrete subgroup of  $SO(n+1, 1)$  which acts discretely and properly discontinuously on  $\mathbb{H}^{n+1}$ . Suppose that the Poincaré exponent  $\delta(\Gamma)$  lies in the interval  $(0, n/2)$ , and let  $\Lambda = \Lambda(\Gamma)$  be the limit set of  $\Gamma$  in  $S^n$ . If  $\dim \Lambda < \frac{1}{2}(n-2\gamma)$ , then  $\Omega = S^n \setminus \Lambda$  admits a complete conformally related metric  $g$  with  $Q_\gamma^g > 0$ .*

**Theorem 4.4.** *Let  $X$  be any  $n$ -manifold with nonnegative Yamabe constant and  $\Lambda$  a  $k$ -dimensional submanifold with  $k < \frac{1}{2}(n-2)$ . Then for all  $\gamma$  in some range  $(1-\epsilon, 1+\epsilon)$ , there exists a solution to*

$$P_\gamma^{\bar{g}} u = Q_\gamma^g u^{\frac{n+2\gamma}{n-2\gamma}}, \quad u > 0.$$

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